18.700 - LINEAR ALGEBRA, DAY 17
ADJOINT, SELF-ADJOINT, AND NORMAL OPERATORS
THE SPECTRAL THEOREM

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I. Pre-class Planning

I.1. **Goals for lesson.**

(1) Students will learn properties of the adjoint of a linear operator.
(2) Students will learn the definition of self-adjoint and normal operators.
(3) Students will learn the statements of the real and complex Spectral Theorems.

I.2. **Methods of assessment.**

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. **Materials to bring.** (1) Laptop + adapter (2) Worksheets (3) Chalk

(0:00)

II. LESSON PLAN

Announcements: • Midterm Exam 2: Wednesday, November 13th in class • Extra office hours: Friday, November 8th, 1 - 2pm. • Exam review session: Nov 12 (Tue) 19:00 - 21:00, 2-361

II.1. **Last time.**

- Proved properties of orthogonal complements.
- Defined orthogonal projection onto a subspace.
- Showed that, given a vector v, the closest point of a subspace U to v is $\text{proj}_U(v)$.
- Defined linear functionals.
- Proved the Riesz Representation Theorem.
- Defined the adjoint of a linear operator.

II.2. **7A: Adjoint, Self-Adjoint, and Normal Operators, cont.**

For today, let V and W be nonzero finite-dimensional inner product spaces over $\mathbb{F}$.

Definition 1. Given $T \in \mathcal{L}(V, W)$, the *adjoint of* T is the unique linear map $T^* \in \mathcal{L}(W, V)$ such that

$$\langle T(v), w\rangle = \langle v, T^*(w)\rangle \qquad \text{(*)}$$

for all $v \in V$ and all $w \in W$.

Proposition 2. *Suppose $T \in \mathcal{L}(V, W)$.*

(i) $(S+T)^* = S^* + T^*$ for all $S \in \mathcal{L}(V, W)$.
(ii) $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbb{F}$.
(iii) $(T^*)^* = T$.
(iv) Let U be a finite-dimensional inner product space. Then $(ST)^* = T^*S^*$ for all $S \in \mathcal{L}(W, U)$.
(v) $I^* = I$.
(vi) If T is invertible, then T^* is also invertible, and $(T^*)^{-1} = (T^{-1})^*$.

Proof. Suppose $v \in v$, $w \in W$, and $\lambda \in \mathbb{F}$.

(i) By definition,

$$\langle (S+T)(v), w\rangle = \langle v, (S+T)^*(w)\rangle .$$

Now

$$\begin{aligned}\langle (S+T)(v), w\rangle &= \langle S(v), w\rangle + \langle T(v), w\rangle = \langle v, S^*(w)\rangle + \langle v, T^*(w)\rangle \\ &= \langle v, S^*(w) + T^*(w)\rangle = \langle v, (S^* + T^*)(w)\rangle .\end{aligned}$$

(ii) Similar.

(iii)

$$\langle T^*(w), v\rangle = \overline{\langle v, T^*(w)\rangle} = \overline{\langle T(v), w\rangle} = \langle w, T(v)\rangle .$$

(iv) Given $S \in \mathcal{L}(W, U)$ and $u \in U$, then

$$\langle (ST)(v), u\rangle = \langle S(T(v)), u\rangle = \langle T(v), S^*(u)\rangle = \langle v, T^*(S^*(u))\rangle .$$

(v) Exercise.

(vi) Apply * to the equations $T^{-1}T = I$ and $TT^{-1} = I$ and then apply the two previous parts.

□

Proposition 3. *Suppose $T \in \mathcal{L}(V, W)$. Then*

(i) $\ker(T^*) = (\mathrm{img}(T))^\perp$;
(ii) $\mathrm{img}(T^*) = \ker(T)^\perp$;
(iii) $\ker(T) = (\mathrm{img}(T^*))^\perp$;
(iv) $\mathrm{img}(T) = (\ker(T^*))^\perp$.

Proof. (i) ($\subseteq$): Given $w \in \ker(T^*)$, then $0 = T^*(w)$. Given $x \in \mathrm{img}(T)$, then $x = T(v)$ for some $v \in V$. Then

$$\langle x, w \rangle = \langle T(v), w \rangle = \langle v, T^*(w) \rangle = \langle v, 0 \rangle = 0 .$$

Thus $w \in (\mathrm{img}(T))^\perp$.
($\supseteq$): Similar.

(ii) Replace T by T^* in the previous part.
(iii) Take the orthogonal complement of (i).
(iv) Take the orthogonal complement of (ii).

□

Q: After having chosen bases, how does the matrix of T^* relate to the matrix of T?

Definition 4. Let $A \in M_{m \times n}(\mathbb{F})$. The *conjugate transpose of A*, denoted A^*, is defined by

$$(A^*)_{ij} = \overline{(A^t)_{ij}} = \overline{A_{ji}} .$$

Proposition 5. *Let $T \in \mathcal{L}(V, W)$, let $\mathcal{E} := (e_1, \ldots, e_n)$, and $\mathcal{F} := (f_1, \ldots, f_m)$ be orthonormal bases for V and W, respectively. Then*

$$_{\mathcal{E}}[T^*]_{\mathcal{F}} = ({}_{\mathcal{F}}[T]_{\mathcal{E}})^* .$$

Proof. Recall that the k^{th} column of $_{\mathcal{F}}[T]_{\mathcal{E}}$ is $[T(e_k)]_{\mathcal{F}}$. Since $\mathcal{F}$ is orthonormal, we have

$$T(e_k) = \langle T(e_k), f_1 \rangle f_1 + \cdots \langle T(e_k), f_m \rangle f_m ,$$

so

$$[T(e_k)]_{\mathcal{F}} = \begin{pmatrix} \langle T(e_k), f_1 \rangle \\ \vdots \\ \langle T(e_k), f_1 \rangle \end{pmatrix} .$$

Thus

$$({}_{\mathcal{F}}[T]_{\mathcal{E}})_{jk} = \langle T(e_k), f_j \rangle .$$

Similarly

$$({}_{\mathcal{E}}[T^*]_{\mathcal{F}})_{jk} = \langle T^*(f_k), e_j \rangle = \langle f_k, T(e_j) \rangle = \overline{\langle T(e_j), f_k \rangle} .$$

Thus

$$({}_{\mathcal{E}}[T^*]_{\mathcal{F}})_{jk} = \overline{({}_{\mathcal{F}}[T]_{\mathcal{E}})_{kj}} = \overline{({}_{\mathcal{F}}[T]_{\mathcal{E}})^t_{jk}} .$$

□

II.2.1. *Self-adjoint operators.*

Definition 6. An operator $T \in \mathcal{L}(V)$ is *self-adjoint* if $T = T^*$.

Lemma 7. *If $\mathcal{E}$ is an orthonormal basis for V, then T is self-adjoint iff $[T^*]_\mathcal{E} = ([T]_\mathcal{E})^*$.*

Example 8. Let $T \in \mathcal{L}(\mathbb{F}^2)$ be the linear operator such that

$$[T]_\mathcal{E} = \begin{pmatrix} 2 & i \\ -i & 7 \end{pmatrix} .$$

T is self-adjoint.

Remark 9. The adjoint of a linear operator is analogous to the complex conjugate of a complex number.

Proposition 10. *Let $T \in \mathcal{L}(V)$ be self-adjoint. Then every eigenvalue of T is real.*

Proof. Suppose $\lambda \in \mathbb{F}$ is an eigenvalue of T, so $T(v) = \lambda v$ for some $0 \neq v \in V$. Then

$$\begin{aligned} \langle T(v), v\rangle &= \langle \lambda v, v\rangle = \lambda < v, v > = \lambda \|v\|^2 \\ \langle T(v), v\rangle &= \langle v, T(v)\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle = \overline{\lambda}\|v\|^2 . \end{aligned}$$

Since $v \neq 0$, then $\|v\|^2 \neq 0$, so $\overline{\lambda} = \lambda$. Thus $\lambda \in \mathbb{R}$. □

Proposition 11. *Suppose V is an inner product space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Then T is self-adjoint iff $\langle T(v), v\rangle \in \mathbb{R}$ for all $v \in V$.*

Proof. ($\Rightarrow$): Assume T is self-adjoint, so $T = T^*$. Given $v \in V$, then

$$\langle T^*(v), v\rangle = \overline{\langle v, T^*(v)\rangle} = \overline{\langle T(v), v\rangle} .$$

Then

$$0 = \langle 0(v), v\rangle = \langle (T - T^*)(v), v\rangle = \langle T(v), v\rangle - \langle T^*(v), v\rangle = \langle T(v), v\rangle - \overline{\langle T(v), v\rangle} .$$

Thus $\langle T(v), v\rangle$ is real.
($\Leftarrow$): Exercise. (Similar.) □

II.2.2. *Normal operators.*

Definition 12. An operator $T \in \mathcal{L}(V)$ is *normal* if T commutes with its adjoint, i.e.,

$$TT^* = T^*T .$$

Example 13. Let $T \in \mathcal{L}(\mathbb{F}^2)$ whose matrix with respect to the standard basis is

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} .$$

Since its matrix with respect to the standard basis (which is orthonormal) is not symmetric, then T is not self-adjoint. However, T is normal. [Compute TT^* and T^*T.]

Proposition 14. *Suppose $T \in \mathcal{L}(V)$. Then T is normal iff $\|T(v)\| = \|T^*(v)\|$ for all $v \in V$.*

Proof. Suppose T is normal. Then $T^*T = TT^*$, so given $v \in V$,

$$\langle TT^*(v), v\rangle = \langle T^*(v), T^*(v)\rangle = \|T^*(v)\|^2$$

$$\langle T^*T(v), v\rangle = \langle T(v), T(v)\rangle = \|T(v)\|^2 .$$

One can show that each of these steps is reversible, so the reverse implication is also true. □

Proposition 15. *Suppose $T \in \mathcal{L}(V)$ is normal. Then*

(i) $\ker(T) = \ker(T^*)$;
(ii) $\operatorname{img}(T) = \operatorname{img}(T^*)$;
(iii) $V = \ker(T) \oplus \operatorname{img}(T)$;
(iv) $\mathbb{T} - \lambda I$ *is normal for all* $\lambda \in \mathbb{F}$;
(v) if $v \in V$ *and* $\lambda \in \mathbb{F}$, *then* $T(v) = \lambda v$ *iff* $T^*(v) = \overline{\lambda} v$.

Proof. (i) Given $v \in V$, then

$$v \in \ker(T) \iff \|T(v)\| = 0 \iff \|T^*(v)\| = 0 \iff v \in \ker(T^*)$$

where the middle equality holds by the previous proposition.

(ii) By a previous result, $\operatorname{img}(T) = (\ker(T^*))^\perp$ and $(\ker(T))^\perp = \operatorname{img}(T^*)$. Then

$$\operatorname{img}(T) = (\ker(T^*))^\perp = (\ker(T))^\perp = \operatorname{img}(T^*)$$

by part (i).

(iii) We have

$$V = (\ker(T)) \oplus (\ker(T))^\perp = \ker(T) \oplus \operatorname{img}(T^*) = \ker(T) \oplus \operatorname{img}(T) .$$

(iv) Exercise.

(v) [Leave as exercise if necessary.] Suppose $v \in V$ and $\lambda \in \mathbb{F}$. By the previous part,

$$\|(T - \lambda I)(v)\| = \|(T - \lambda I)^*(v)\| = \|(T^* - \overline{\lambda} I)(v)\| .$$

Thus $T(v) = \lambda v$ iff $T^*(v) = \overline{\lambda} v$.

□

Proposition 16. *Suppose $T \in \mathcal{L}(V)$ is normal. Then the eigenvectors of T with distinct eigenvalues are orthogonal.*

Proof. [Leave as exercise if necessary.] Suppose $\alpha \neq \beta$ are eigenvalues of T with corresponding eigenvectors u and v, so $T(u) = \alpha u$ and $T(v) = \beta v$. Then

$$\begin{aligned} 0 &= \langle T(u), v\rangle - \langle T(u), v\rangle = \langle T(u), v\rangle - \langle u, T^*(v)\rangle = \langle \alpha u, v\rangle - \langle u, \overline{\beta} v\rangle \\ &= \alpha\langle u, v\rangle - \beta\langle u, v\rangle = (\alpha - \beta)\langle u, v\rangle . \end{aligned}$$

Since $\alpha - \beta \neq 0$, then $\langle u, v\rangle = 0$. □

II.3. **7B The Spectral Theorem.** Let V be a finite-dimensional $\mathbb{C}$-vector space and $T \in \mathcal{L}(V)$.

- There exists a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is upper triangular.
- If T is diagonalizable, then there exists a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal. In this case, $\mathcal{B}$ consists of eigenvectors of T.

Q: Now let V be a finite-dimensional inner product space. When does V have an *orthonormal* basis $\mathcal{E}$ consisting of eigenvectors of T?

A:

- For $\mathbb{F} = \mathbb{C}$, when T is normal.
- For $\mathbb{F} = \mathbb{R}$, when T is self-adjoint.

Throughout today, let V be a finite-dimensional inner product space over $\mathbb{F}$.

Theorem 17 (Spectral Theorem over $\mathbb{C}$)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. TFAE.*

(i) T is normal.
(ii) There is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal.
(iii) There is an orthonormal basis of V consisting of eigenvectors of T.

Proof. We have already seen (b) $\iff$ (c), so it remains to show (a) $\iff$ (b).

(a) $\implies$ (b): Assume T is normal. Since minpoly(T) splits into degree 1 factors, then there is an orthonormal basis $\mathcal{E} := (e_1, \dots, e_n)$ of V such that $[T]_{\mathcal{E}}$ is upper triangular.

$$[T]_{\mathcal{E}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix} .$$

Since $\mathcal{E}$ is orthonormal, then

$$[T^*]_{\mathcal{E}} = ([T]_{\mathcal{E}})^* = \begin{pmatrix} \overline{a_{11}} & & & \\ \overline{a_{12}} & \overline{a_{22}} & & \\ \vdots & \vdots & \ddots & \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{pmatrix} .$$

We will show that $[T]_{\mathcal{E}}$ is diagonal. Then

$$T(e_1) = a_{11} e_1$$
$$T^*(e_1) = \overline{a_{11}} e_1 + \overline{a_{12}} e_2 + \cdots + \overline{a_{1n}} e_n .$$

Since $e_1, \dots, e_n$ is orthonormal, then

$$\begin{aligned} \|T(e_1)\|^2 &= |a_{11}|^2 \\ \|T^*(e_1)\|^2 &= |\overline{a_{11}}|^2 + |\overline{a_{12}}|^2 + \cdots + |\overline{a_{1n}}|^2 \\ &= |a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2 \end{aligned}$$

by the Pythagorean Theorem. Since T is normal, then these are equal by a previous result. Subtracting, then

$$0 = |a_{12}|^2 + \cdots + |a_{1n}|^2$$

so $0 = a_{12} = \cdots = a_{1n}$. [Update matrices for $[T]_{\mathcal{E}}$ and $[T^*]_{\mathcal{E}}$ by filling in 0s.]

Now we have

$$\begin{aligned} T(e_2) &= a_{22}e_2 \\ T^*(e_2) &= \overline{a_{22}}e_2 + \cdots \overline{a_{2n}}e_n \end{aligned}$$

so

$$\begin{aligned} \|T(e_2)\|^2 &= |a_{22}|^2 \\ \|T^*(e_2)\|^2 &= |\overline{a_{22}}|^2 + |\overline{a_{23}}|^2 + \cdots + |\overline{a_{2n}}|^2 \\ &= |a_{22}|^2 + |a_{23}|^2 + \cdots + |a_{2n}|^2 \,. \end{aligned}$$

By analogous reasoning, then $0 = a_{23} = \cdots = a_{2n}$. Proceeding similarly, we find that $a_{ij} = 0$ for all $i \neq j$. Thus $[T]_{\mathcal{E}}$ is diagonal.

(b) $\implies$ (a): Assume there is an orthonormal basis $\mathcal{E}$ of V such that $[T]_{\mathcal{E}}$ is diagonal. Then $[T^*]_{\mathcal{E}} = ([T]_{\mathcal{E}})^*$, which is also diagonal. Since diagonal matrices commute, then

$$[TT^*]_{\mathcal{E}} = [T]_{\mathcal{E}}[T^*]_{\mathcal{E}} = [T^*]_{\mathcal{E}}[T]_{\mathcal{E}} = [T^*T]_{\mathcal{E}} \,,$$

so $TT^* = T^*T$. (Recall that $[\cdot]_{\mathcal{E}} : \mathcal{L}(V) \to M_{n\times n}(\mathbb{F})$ is an isomorphism.) Thus T is normal. $\square$

Theorem 18 (Spectral Theorem over $\mathbb{R}$)**.** *Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. TFAE.*

(i) T is self-adjoint.

(ii) There is an orthonormal basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal.

(iii) There is an orthonormal basis of V consisting of eigenvectors of T.

We'll need some preliminary results to prove the theorem.

Lemma 19. *Suppose $T \in \mathcal{L}(V)$ is self-adjoint, and $b, c \in \mathbb{R}$ with $b^2 - 4c < 0$. Then*

$$T^2 + bT + cI$$

is an invertible operator.

Skip if necessary. Suppose $0 \neq v \in V$. By Cauchy-Schwarz, we have

$$\begin{aligned} &|\langle bT(v), v\rangle| \leq \|bT(v)\|\|v\| = |b|\|T(v)\|\|v\| \\ &\iff -|b|\|T(v)\|\|v\| \leq \langle bT(v), v\rangle \leq |b|\|T(v)\|\|v\| \,. \end{aligned}$$

Since T is self-adjoint, then

$$\begin{aligned} \langle (T^2 + bT + cI)(v), v\rangle &= \langle T^2(v), v\rangle + b\langle T(v), v\rangle + c\langle v, v\rangle \\ &= \langle T(v), T(v)\rangle + b\langle T(v), v\rangle + c\|v\|^2 \\ &\geq \|T(v)\|^2 - |b|\langle T(v), v\rangle + c\|v\|^2 \\ &= \left(\|T(v)\| - \frac{|b|\|v\|}{2}\right)^2 - \frac{|b|^2\|v\|^2}{4} + c\|v\|^2 \\ &= \underbrace{\left(\|T(v)\| - \frac{|b|\|v\|}{2}\right)^2}_{\geq 0} + \underbrace{\frac{4c - b^2}{4}}_{>0} \|v\| > 0 \,. \end{aligned}$$

Thus $(T^2 + bT + cI)(v) \neq 0$. This shows that $\ker(T^2 + bT + cI) = \{0\}$, so this operator is injective. Since the domain and codomain are both V, then this implies that it is invertible. □

Proposition 20. *Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Then*

$$\text{minpoly}(T) = (z - \lambda_1) \cdots (z - \lambda_m)$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$.

Proof. <u>Case 1</u>: $\mathbb{F} = \mathbb{C}$. Then minpoly(T) splits into degree 1 factors. Recall that the roots of minpoly(T) are the eigenvalues of T. Since T is self-adjoint, by a previous result, all its eigenvalues are real. Thus minpoly has the desired form.

<u>Case 2</u>: $\mathbb{F} = \mathbb{R}$. Then minpoly(T) factors into a product of degree 1 and degree 2 factors:

$$\text{minpoly}(T) = (z - \lambda_1) \cdots (z - \lambda_m)(z^2 + b_1 z + c_1) \cdots (z^2 + b_N z + c_N)$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and some $b_1, \ldots, b_N, c_1, \ldots, c_N \in \mathbb{R}$ with $b_k^2 - 4c_k < 0$ for all $k = 1, \ldots, N$. <u>Goal</u>: $N = 0$. Since this is minpoly(T), then

$$(T - \lambda_1) \cdots (T - \lambda_m)(T^2 + b_1 T + c_1) \cdots (T^2 + b_N T + c_N) = 0\,. \qquad (*)$$

For contradiction, suppose $N > 0$. Then $(T^2 + b_N T + c_N)$ is invertible by the previous result, so multiplying both sides of $(*)$ by its inverse, we have

$$(T - \lambda_1) \cdots (T - \lambda_m)(T^2 + b_1 T + c_1) \cdots (T^2 + b_{N-1} T + c_{N-1}) = 0\,.$$

But this has degree strictly smaller than minpoly(T), contradiction. Thus $N = 0$. □

Proof of Spectral Theorem over $\mathbb{R}$. We have already seen (b) $\iff$ (c).

(a) $\implies$ (b): Assume T is self-adjoint. Since minpoly(T) splits into degree 1 factors by the previous result, then there exists an orthonormal basis $\mathcal{E}$ of V such that $[T]_{\mathcal{E}}$ is upper triangular. Since T is self-adjoint, then

$$([T]_{\mathcal{E}})^t = ([T]_{\mathcal{E}})^* = [T^*]_{\mathcal{E}} = [T]_{\mathcal{E}}\,.$$

Now $([T]_{\mathcal{E}})^t$ is lower triangular, so we must have that $[T]_{\mathcal{E}}$ is diagonal.

(b) $\implies$ (a): Assume there exists an orthonormal basis $\mathcal{E}$ of V such that $[T]_{\mathcal{E}}$ is diagonal. Then

$$[T]_{\mathcal{E}} = ([T]_{\mathcal{E}})^t = ([T]_{\mathcal{E}})^*$$

so $T = T^*$. Thus T is self-adjoint. □