MATH 18.700, DAY 1
SYSTEMS OF LINEAR EQUATIONS

SAM SCHIAVONE

CONTENTS

I. PRE-CLASS PLANNING

I.1. Goals for lesson.

(1) Students will understand expectations for the course.
(2) Students will understand the definition of a system of linear equations.
(3) Students will learn the elementary row operations.
(4) Students will learn how to row reduce a matrix.
(5) Students will learn how to write solutions in parametric form.

I.2. Methods of assessment.

(1) Student responses to questions posed during lecture
(2) Student responses to questions posed during lecture

I.3. Materials to bring.
(1) Laptop + adapter (2) Colored chalk

II. Lesson Plan

(0:00)

Hi everyone and welcome to 18.700: Linear Algebra. I'm Sam. I'm a postdoc here at MIT and I'll be your instructor for this course. [Name game. Remind students about online survey.]

(0:05)

II.1. **Organizational details.** Before we talk about math, I'd like to cover some organizational details and show you the course website. [Show the website. Mention Canvas. Mention possible office hours: Monday 1 - 2pm, Tuesday 2 - 3pm. Ask if there are conflicts for the midterm exams. Mention that first homework assignment will be posted later today, due next Wednesday.]

(0:10)

II.2. **Why linear algebra?** Motivating question: How do we solve systems of equations?

Answer:

- Arbitrary equations? Hard! Numerical analysis provides approximate solutions.
- Polynomial equations? Still hard! Computational algebraic geometry and Gröbner bases can sometimes provide solutions, but as the number of variables and degree increases, these methods become impractical.
- Linear equations, i.e., polynomial equations of degree 1. This is what we'll study.

Roughly speaking, a linear system is one in which the output is proportional to the input: an input twice as large yields an output twice as large. Linear systems are very well understood thanks to the success of the theory of linear algebra. Even when considering a nonlinear system, we often approximate it with a linear one. One example of this idea of linear approximation is the derivative in calculus.

Linear algebra is also used in virtually every branch of math: graph theory, functional analysis, differential geometry, algebraic geometry, algebraic number theory, differential equations. It is also used in virtually every area in science.

(0:15)

II.3. **What is linear algebra?** Short answers:

- The study of solutions sets to systems of linear equations and the transformations of these solution sets.
- The study of finite-dimensional vector spaces and linear maps.

Notation: Unless otherwise mentioned, throughout this course, the symbol $\mathbb{F}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$, the sets of real or complex numbers, respectively. (We use $\mathbb{F}$ because $\mathbb{R}$ and $\mathbb{C}$ are examples of *fields*, algebraic structures in which one can add, subtract, multiply, and divide.

Definition 1. A *linear equation* in the variables $x_1, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + \cdots + a_nx_n = b$$

for some coefficients $a_1, \ldots, a_n, b \in \mathbb{F}$. In other words, the lefthand side must be a polynomial of degree 1 in $x_1, \ldots, x_n$.

For instance, $4x_1x_2 + x_3 = 7$ and $\sqrt{x_1} - 7x_2 = 0$ are not linear. [Ask students: why?]

Definition 2.

- A *system of linear equations* or *linear system* is a collection of one or more linear equations (in the same variables). [Give example]

- A *solution* to a linear system is a list $(s_1, \ldots, s_n)$ of numbers that satisfy each equation, i.e., that make each equation true when $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$.
- The *solution set* is the set of all possible solutions. Two linear systems are *equivalent* if they have the same solution set.

(0:25)

Example 1. Intersection of two lines.

$$\begin{array}{rrcl} x_1 & -x_2 & = & 0 \\ 2x_1 & +x_2 & = & 3 \end{array}$$

[Draw picture of lines.] The (unique) solution is $(1,1)$. An equivalent system is

$$\begin{array}{rrcl} x_1 & & = & 1 \\ & x_2 & = & 1 \end{array}$$

[Ask students: Can you think of two lines that have no points of intersection? What does the picture look like?] There is one last possibility, illustrated by the following linear system:

$$\begin{array}{rrcl} x_1 & -x_2 & = & 1 \\ 2x_1 & -2x_2 & = & 2 \end{array}$$

These two equations define the same line, so every point on this line is a solution. Thus there are infinitely many solutions.

Remark 1. This is a phenomenon that holds in general for linear systems: a linear system has either

(1) exactly one solution,
(2) infinitely many solutions, or
(3) no solutions.

We will prove this later.

Definition 3. A linear system is *consistent* if it has a solution (at least one). If it has no solutions, it is *inconsistent*.

(0:35)

II.4. **Matrix notation.** Rewriting all the variables in a linear system can be tedious. We can instead record the relevant information more compactly in a *matrix* (a rectangular array of numbers) as follows. Consider the following linear system.

$$\begin{array}{rrrcr} -2x_1 & +2x_2 & +10x_3 & = & 2 \\ -3x_1 & +x_2 & +9x_3 & = & 5 \\ 4x_1 & -2x_2 & -14x_3 & = & -6 \end{array}$$

The *coefficient matrix* of the system is

$$\begin{pmatrix} -2 & 2 & 10 \\ -3 & 1 & 9 \\ 4 & -2 & -14 \end{pmatrix}$$

The i,j entry is the coefficient of x_i in the j^{th} equation of the system. The *augmented matrix* of the system is

$$\left(\begin{array}{rrr|r} -2 & 2 & 10 & 2 \\ -3 & 1 & 9 & 5 \\ 4 & -2 & -14 & -6 \end{array}\right)$$

where the last column contains the constants from the righthand side of the equations.

(0:40)

II.5. **Solving a linear system.** [Switch to slides. Do an example of 3×3 example that has 1-dimensional solution space.]

$$\begin{array}{rrrcr} -2x_1 & +2x_2 & +10x_3 & = & 2 \\ -3x_1 & +x_2 & +9x_3 & = & 5 \\ 4x_1 & -2x_2 & -14x_3 & = & -6 \end{array}$$

$$\left(\begin{array}{rrr|r} -2 & 2 & 10 & 2 \\ -3 & 1 & 9 & 5 \\ 4 & -2 & -14 & -6 \end{array}\right) \rightsquigarrow \left(\begin{array}{rrr|r} 1 & -1 & -5 & -1 \\ -3 & 1 & 9 & 5 \\ 4 & -2 & -14 & -6 \end{array}\right) \rightsquigarrow \left(\begin{array}{rrr|r} 1 & -1 & -5 & -1 \\ 0 & -2 & -6 & 2 \\ 0 & 2 & 6 & -2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{rrr|r} 1 & -1 & -5 & -1 \\ 0 & -2 & -6 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{rrr|r} 1 & -1 & -5 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{rrr|r} 1 & 0 & -2 & -2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t-2 \\ -3t-1 \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} t + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

[Ask students what geometric shape a given equation looks like. Hint: should be familiar from multivariable calc. Show picture of the planes intersecting in a line.]

In solving this system, we employed three basic operations, called *elementary row operations*:

(1:00) Elementary Row Operations:

(1) (Add a multiple): Add a multiple of one row to another
(2) (Swap) Swap two rows
(3) (Rescale) Multiply all entries in a row by a nonzero constant.

We used these row operations to solve the above linear system, a process called *row reduction* or *Gaussian elimination*. We will explore this process in greater detail shortly.

Two important questions about a linear system:

1) Does the system have a solution, i.e., is it consistent?
2) If so, is the solution unique? How many solutions are there?

(1:05)

II.6. **Echelon forms.** We saw an example of how we can use elementary row operations to take a given linear system and produce an equivalent but simpler linear system. Now we will formally write down this algorithm.

Definition 4.

- A row of a matrix is *nonzero* if at least one entry in the row is not zero. We define nonzero columns similarly.
- The *leading entry* of a nonzero row is the left-most nonzero entry.

Definition 5.

- A matrix is in *echelon form* (or *row echelon form*) if it satisfies the following three properties:
  - All nonzero rows are above any rows of all zeros. (rows of all zeros are at the bottom)
  - Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  - All entries in a column below a leading entry are zero.
- A matrix is in *reduced echelon form* (or *reduced row echelon form*), often abbreviated RREF, if it is in echelon form and satisfies two additional conditions:
  - The leading entry in each nonzero row is a 1.
  - Each leading 1 is the only nonzero entry in its column.

[Give examples and non-examples.]

(1:15)

Theorem (Uniqueness of reduced echelon form)**.** *Every matrix is row equivalent to a* unique *reduced echelon matrix.*

Remark 2. Although an echelon form of a matrix is not unique, the positions of the leading entries are uniquely determined.

Definition 6.

- A *pivot position* in a matrix A is a location in A corresponding to a leading 1 in the reduced echelon form for A.
- A *pivot column* is a column of A that contains a pivot position.
- A *pivot* is a nonzero entry in a pivot position.

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II.7. **Row reduction algorithm.** The row reduction algorithm consists of 5 steps:

(1) Take the leftmost nonzero column. This is the leftmost pivot column. Make the pivot position at the top of the column.
(2) Select any nonzero entry in the leftmost pivot column as a pivot and move it into the pivot position (top) interchanging rows as necessary.
(3) Use row operations to make all entries under the pivot zero.
(4) Consider the submatrix obtained by removing the row and column containing the pivot. Go to the first step and repeat until there is no submatrix left to consider.
(5) Use row operations to create zeros above each pivot.

[Label (1) - (4) as echelonization, (5) as reduction. Also, DON'T ERASE THE ALGORITHM!]

(1:25)

Example 2. Let's row reduce the following augmented matrix. [Ask students. Explicitly point out the various steps during the example.]

$$\left(\begin{array}{ccc|c} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{array}\right)$$

(Should get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

for the RREF.)

Remark 3. There may be many ways to obtain the RREF of a matrix.

(1:30)

II.8. **Worksheet.**

(1:40)

II.9. **Parametric form for solutions.** Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This corresponds to the linear system

$$\boxed{\begin{array}{rrl} x_1 & +3x_3 & = 1 \\ x_2 & -2x_3 & = 5 \end{array}} \longrightarrow \boxed{\begin{array}{l} x_1 = 1 - 3x_3 \\ x_2 = 5 + 2x_3 \end{array}}$$

The variables corresponding to pivot columns (i.e x_1 and x_2) are called *basic* or *leading variables*.

The other variables (i.e., x_3) are called *free variables*.

We can use this description to write the set of solutions parametrically:

$$\begin{cases} x_1 = 1 - 3x_3 \\ x_2 = 5 + 2x_3 \\ x_3 \text{ free} \end{cases}$$

We call this the *parametric description* of the solutions. We can express this in vector notation as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-3t \\ 5+2t \\ t \end{bmatrix} = \begin{bmatrix} -3t \\ 2t \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}.$$

[Ask students how many solutions this system has.]

(1:45)

II.10. **Worksheet part 2.**

(1:50)

II.11. **Existence and uniqueness of solutions.**

Theorem.

(1) *A linear system is consistent iff the right most column of the augmented matrix is* not *a pivot column, i.e., iff an echelon form for the matrix does* not *contain a row of the form*

$$\begin{bmatrix} 0 & \cdots & 0 & | \; b \end{bmatrix}$$

with b nonzero.

(2) *If the linear system is consistent, then the solution set contains either*
  (i) *a unique solution, if there are no free variables*
  (ii) *infinitely many solutions, if there is one or more free variables.*

[Remark on proofs briefly.]

(1:55)

II.12. **Summary.** Solving a linear system via row reduction:

(1) Write the augmented matrix of the system.
(2) Use the row reduction algorithm to put the matrix in echelon form. If the matrix is inconsistent, stop; otherwise continue to the next step.
(3) Keep row reducing to put the matrix in reduced echelon form.
(4) Write the linear system corresponding to the reduced echelon form.
(5) Express the solution to the system in parametric form.