Recitation 01: Finite Automata, Regular Languages

In this recitation, we will review finite automata and see several examples of how to construct them to recognize specific languages. We will also review regular languages and regular operations, concluding with two methods of proving that all finite languages are regular.

Logistics

Recitations will primarily be for going over lecture material in more detail and working through exercises, as opposed to teaching supplemental material. They are recommended if you find yourself struggling with the course content. Attendance is not mandatory, but TAs will make note of active participation, which may be taken into account if you end up near a grade boundary.

The primary content of each recitation is the same; you do not need to attend multiple (and you will not get double the participation score). You are encouraged to attend the same recitation consistently so that TAs can learn your name, though you are free to switch at any point. Make sure to sign in at each recitation so that your attendance can be counted.

Recitation notes will be published shortly after each recitation; they may differ slightly from the content of your recitation due to variations in different instructor sections.

Finite Automata

A *finite automaton* is a model of computation consisting of finitely many states and transitions. In particular, the states include exactly one *start state* and zero or more *accept states*. The strings inputted into a finite automaton consist of symbols from a specified alphabet, often $\{0, 1\}$. The formal definition of a *deterministic finite automaton* (or *DFA*), expressed as a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, was described in Lecture 1.

We'll see several models of computation throughout the course; finite automata are among the simplest because the number of states is *finite*.

To review the formal definition of a finite automaton, see Definition 1.5 in the textbook.

The *language* of a finite automaton is the set of all strings that it accepts. Formally, a DFA $M = (Q, \Sigma, \delta, q_0, F)$ *accepts* a string $w =$

$w_1 w_2 \cdots w_n$, where $w_i \in \Sigma$, if $\exists\ r_0, r_1, ..., r_n \in Q$ (there exists a series of states) such that

1. $r_0 = q_0$ (the initial state is the starting state)
2. $r_i = \delta(r_{i-1}, w_i)$ for all $i = 1, ..., n$ (the transitions are valid)
3. $r_n \in F$ (the final state is an accept state).

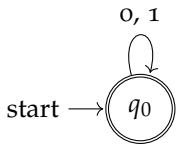
Exercise 1. *Construct a DFA that recognizes each of the following languages, where the alphabet is* $\Sigma = \{0, 1\}$*:*

1. Σ^*
2. $\emptyset$
3. $\{\varepsilon\}$

Remember that Σ^* is the set of all strings consisting of zero or more 0s and 1s.

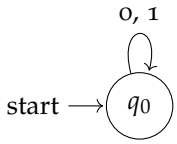
Solution. We construct DFAs for each of the three languages.

1. Σ^*



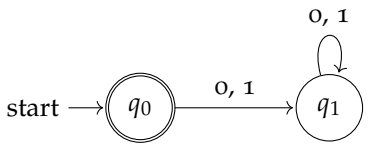
The only state is both the start state and the only accept state. This means that any string that enters the DFA will be accepted!

2. $\emptyset$



A small modification from the above: now no strings should be accepted, so we simply remove the accept state. *Note:* any valid DFA with no accept states would also work.

3. $\{\varepsilon\}$



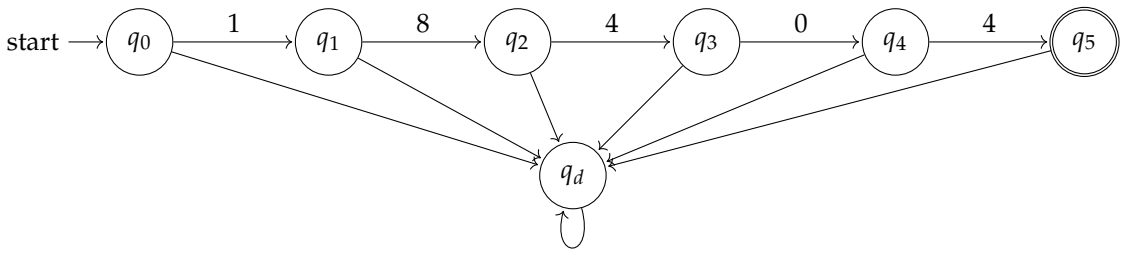
The start state is an accept state. All strings consisting of at least one 0 or 1 will lead us to q_1 and never return to q_0, so they will be rejected. *Note:* any DFA accepting the empty string must have the start state be an accept state.

Exercise 2. *Construct a DFA that recognizes each of the following languages with the given* Σ*:*

1. $\Sigma = \{0, 1, ..., 9\}$, $A_{18404} = \{18404\}$.
2. $\Sigma = \{a, b\}$, $A_{even} = \{w \mid w \text{ has an even number of a's}\}$.

Solution. We construct DFAs for each of the two languages.

1. $\{18404\}$



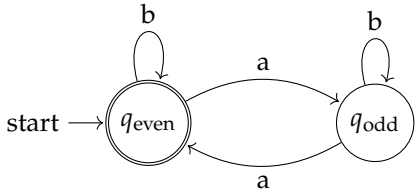
An arrow with no label implies that all alphabet members not included in any other transition from that state will follow this arrow. For example, from $\delta(q_2, n) = q_d$ for all $n \neq 4$.

Here, q_d represents the "*dead state*." Once we land in this state, we will never leave, so the input will not be accepted. When constructing DFAs, making a dead state is useful when we want to never accept if a certain condition occurs (or fails to occur).

Note that q_1 in Exercise 1.3 is also a dead state!

2. $\{w \mid w$ has an even number of a's$\}$

Being in q_{even} represents that we have read in an even number of a's; being in q_{odd} represents that we have read in an odd number of a's.



When constructing DFAs, it's useful to think of each state as a *possibility* for the value that the machine cares about. For example, in Exercise 2.2 above, the value that the machine cares about is whether there is an even or odd number of a's so far: in other words, the *parity* of the number of a's. Since there are two possible parities, we have two states.

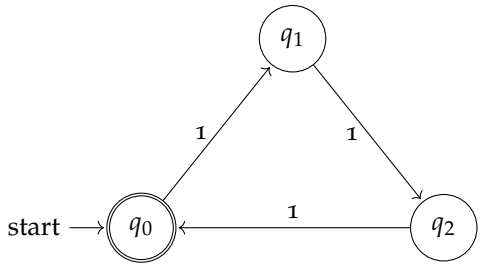
Exercise 3. *With $\Sigma = \{1\}$ (unary strings), construct a DFA that recognizes $A_3 = \{w \mid w$ has length a multiple of 3$\}$. Then, by generalizing this idea, provide a formal definition of a DFA recognizing $A_n = \{w \mid w$ has length a multiple of $n\}$.*

Hint. What is the value that the machine needs to keep track of? How many possibilities are there for that value?

Solution. We know that the DFA should keep track of a value related to the length of the input; however, we don't care about the actual length as much as whether it is divisible by 3. Thus, we create three states to represent the *length modulo 3* of the input so far.

"Length modulo 3" is the remainder when we divide the length by 3.

We start at q_0, representing that the current length is 0 mod 3. Note that this is also the accept state, since we want to accept strings with length 0 mod 3. On the next 1, the length becomes 1 mod 3, then 2 mod 3, then back to 0 mod 3, etc.



For arbitrary n, we want a cycle like the above, but consisting of n states, representing the length being $0, 1, ..., n-1$ modulo n. We want each state to transition to the next in the cycle, and the starting state (corresponding to $0 \mod n$) to be the accept state. $M = (Q, \Sigma, \delta, q_0, F)$ recognizes A_n, where

$$
\begin{aligned}
Q &= \{q_0, ..., q_{n-1}\} \\
\Sigma &= \{1\} \\
\delta(q_i, 1) &= \begin{cases} q_0 & i = n-1 \\ q_{i+1} & \text{else} \end{cases} \\
q_0 &= q_0 \\
F &= \{q_0\}
\end{aligned}
$$

As there is only one symbol, 1, in the alphabet, we only need to define the transition function δ for that symbol.

Regular Languages

A *regular language* is any set of strings that is recognized by some finite automaton.

Let $L_1 = \{a, b\}$, $L_2 = \{b, c\}$. In Lecture 1, we defined the following *regular operations*:

- Union ($\cup$)

  Example: $L_1 \cup L_2 = \{a, b, c\}$.

- Concatenation ($\circ$)

  Example: $L_1 \circ L_2 = L_1 L_2 = \{ab, ac, bb, bc\}$.

- Star ($*$)

  Examples: $\emptyset^* = \{\varepsilon\}$, $L_1^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, ...\}$.

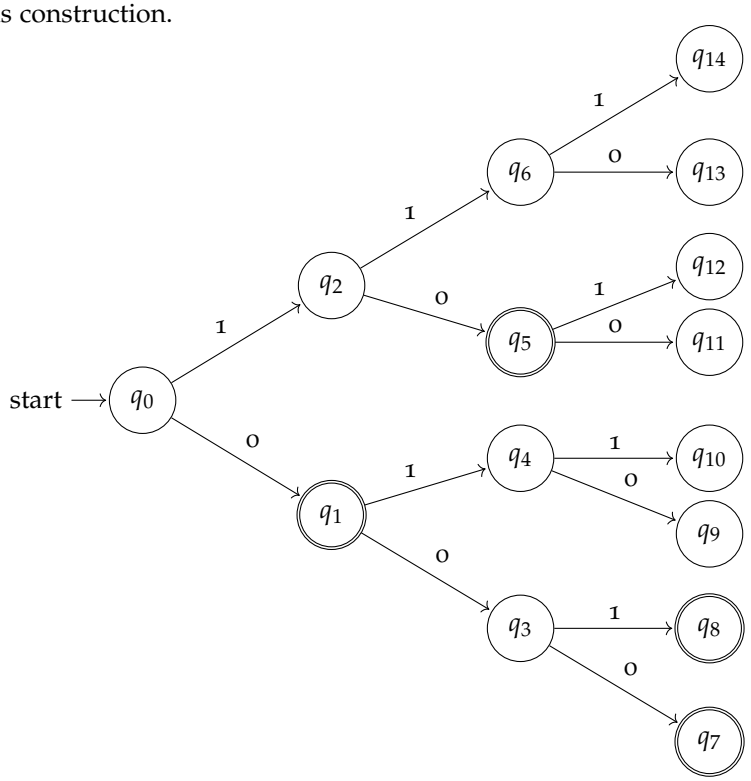
Theorem 1. *Every finite language is regular.*

Proof 1 (construction). Let $L = \{w_1, w_2, ..., w_n\}$ be a finite language. Define k to be the maximum length of a string in L, or $k = \max_{1 \le i \le n} |w_i|$. Construct a tree of depth $k+1$, where each state splits into two new states (one for 0 and one for 1). See the example below for concreteness. Note that there is now a unique path for every string of length k.

Note that an infinite language would have arbitrarily long strings, so it would not be possible to find k. This construction only applies for finite languages.

For each $w_i \in L$, add an accept state corresponding to the state reached after processing w_i. This ensures that strings lead to an accept state if and only if the string is a member of L. Thus, this construction recognizes L, which shows that L is regular. □

For example, the following is the DFA for $L = \{0, 10, 000, 001\}$ using this construction.



In addition, there is an dead state from which states $q_7, ..., q_{14}$ lead to on any input, since we don't want to accept any strings of length longer than 3. We haven't drawn the dead state or its associated transitions to simplify the diagram.

Proof 2 (closure under union). Again let $L = \{w_1, w_2, ..., w_n\}$ be a finite language. Let $L_i := \{w_i\}$ for $i = 1, ..., n$. For each i, we can construct a DFA that recognizes L_i in the same manner as Exercise 2.1 (A_{18404}). Thus, L_i is a regular language. In Lecture 1, we showed that regular languages are closed under union. We know that $L = L_1 \cup L_2 \cup \cdots \cup L_n$, so L is also regular. □

Note that closure under union does not hold for an infinite number of languages, so this proof also only works for finite languages.