1. Let $\text{MODEXP} = \{\langle a, b, c, p \rangle | a, b, c, p \text{ are positive binary integers such that } a^b \equiv c \pmod{p} \}$. Show that $\text{MODEXP} \in \text{P}$. (Hint: Try $b = 8$. You can assume that basic arithmetical operations, such as $+, \times$, and mod, are computable in polynomial time.)

2. Let $\text{UNARY-SSUM}$ be the subset sum problem in which all numbers are represented in unary, i.e., $1^k$ represents the number $k$. Why does the NP-completeness proof for $\text{SUBSET-SUM}$ (see textbook) fail to show $\text{UNARY-SSUM}$ is NP-complete? Show that $\text{UNARY-SSUM} \in \text{P}$.

3. Show that if $\text{P} = \text{NP}$, then every language $A \in \text{P}$, except $A = \emptyset$ and $A = \Sigma^*$, is NP-complete.

4. Show that if $\text{P} = \text{NP}$, we can factor integers in polynomial time. (Note: The algorithm you are asked to provide computes a function, and NP contains languages, not functions. Therefore, you cannot solve this problem simply by saying “factoring is in NP and $\text{P} = \text{NP}$ so factoring is in $\text{P}$”. The assumption $\text{P} = \text{NP}$ implies that all languages in NP are in P, so you need to find an NP language that relates to the factoring function.)

5. Let $\text{SET-SPLITTING} = \{\langle S, C \rangle | S \text{ is a finite set and } C = \{C_1, \ldots, C_k\} \text{ is a collection of subsets of } S, \text{ where the elements of } S \text{ can be colored red or blue so every } C_i \text{ has at least one red element and at least one blue element} \}$. Show that $\text{SET-SPLITTING}$ is NP-complete.

6. In three-valued logic (TVL) we may assign values T, F, or X to variables. A TVL-clause has the form $(x, y) \neq (u, v)$ where $x$ and $y$ are variables and $u, v \in \{T, F, X\}$. A TVL-formula is a collection of TVL-clauses. An TVL-assignment of T, F, or X to the variables satisfies a TVL-clause if it doesn’t violate the inequality, i.e., the pair $(x, y)$ must not equal the pair $(u, v)$ in the assignment. It satisfies a TVL-formula $\phi$ if it satisfies all of $\phi$’s TVL-clauses. Let $\text{TVL-SAT} = \{\langle \phi \rangle | \text{TVL-formula } \phi \text{ has a satisfying TVL-assignment} \}$. Show that $\text{TVL-SAT}$ is NP-complete.

7* (Optional) This problem investigates resolution, a method for proving the unsatisfiability of cnf-formulas. Let $\phi = C_1 \land C_2 \land \cdots \land C_m$ be a formula in cnf, where the $C_i$ are its clauses. Let $C = \{C_i | C_i \text{ is a clause of } \phi \}$. In a resolution step, we take two clauses $C_a$ and $C_b$ in $C$, which both have some variable $x$, where $x$ occurs positively in one of the clauses and negatively in the other. Thus, $C_a = (x \lor y_1 \lor y_2 \lor \cdots \lor y_k)$ and $C_b = (\overline{x} \lor z_1 \lor z_2 \lor \cdots \lor z_l)$, where the $y_i$ and $z_i$ are literals. We form the new clause $(y_1 \lor y_2 \lor \cdots \lor y_k \lor z_1 \lor z_2 \lor \cdots \lor z_l)$ and remove repeated literals. Add this new clause to $C$. Repeat the resolution steps until no additional clauses can be obtained. If the empty clause ( ) is in $C$, then declare $\phi$ unsatisfiable.

Say that resolution is sound if it never declares satisfiable formulas to be unsatisfiable. Say that resolution is complete if all unsatisfiable formulas are declared to be unsatisfiable.

(a) Show that resolution is sound and complete.

(b) Use part (a) to show that $2\text{SAT} \in \text{P}$.