18.404/6.5400 Michael Sipser
Theory of Computation
Due via Gradescope at noon sharp, Thursday, October 5, 2023 **Problem Set 2**

Read all of Chapters 3 and 4.

0.1 Read and solve, <u>but do not turn in</u>: Book, 2.16 . [CFLs closed under $\cup$, $\circ$, *]
Solve by using both CFGs and PDAs.

0.2 Read and solve, <u>but do not turn in</u>: Book, 2.18 . [(CFL $\cap$ regular) is a CFL]
Note, problems marked with $^{\mathsf{A}}$ have solutions in the book.

0.3 Read and solve, <u>but do not turn in</u>: Book, 2.26 . [Chomsky normal form]

0.4 Read and solve, <u>but do not turn in</u>: Book, 2.30c . [CFL Pumping lemma]

1. Let $B = \{\mathtt{a}^i\mathtt{b}^j |\ i \neq j \text{ and } 2i \neq j\}$. Show that B is a CFL. (Hint: think about B differently.) If you give a CFG, explain how it works. If you give a PDA, only describe it informally.

2. Let $\Sigma = \{\mathtt{0}, \mathtt{1}, \mathtt{\#}\}$. Let $C = \{w\mathtt{\#}w^{\mathcal{R}}\mathtt{\#}w |\ w \in \{\mathtt{0},\mathtt{1}\}^*\}$. ($w^{\mathcal{R}}$ is w written in reverse)
   (a) Prove that C is not a CFL.
   (b) Use part (a) to show the class of CFLs is not closed under intersection or complement.
   (c) Let $D = C \cup \Sigma(\Sigma\Sigma\Sigma)^*$. Use part (a) to show that D is not a CFL.

3. A ***useless state*** in a pushdown automaton is never entered on any input string on any computation thread. Consider the problem of determining whether a pushdown automaton has any useless states. Formulate this problem as a language and show that it is decidable. (Hint: Use a theorem from lecture to give a short solution.)

4. Upon receiving your deterministic, single-tape Turing machine back from the Turing Machine Repair Service, you find to your dismay that when it tries to write a symbol on any tape cell, it is only able to squirt a blot of ink on that cell, or it may leave the symbol it read unchanged. When it reads a blot, it treats it as a new symbol ■. Fortunately, you recall hearing somewhere that this does not diminish the class of recognized languages. All you need now is the proof... (Hint: Use lots of tape.)

5. Show that a language is decidable iff some enumerator enumerates the language in string order. Make sure that your solution works even when the enumerator's language is finite. (***String order*** is the standard length-increasing, lexicographic order, see text p 14).

6. Let C be a language. Prove that C is Turing-recognizable iff a decidable language D exists such that $C = \{x |\ \exists y \in \{\mathtt{0},\mathtt{1}\}^*\ (\langle x, y\rangle \in D)\}$. (Hint: You must prove both directions of the "iff". The ($\longleftarrow$) direction is easier. For the ($\longrightarrow$) direction, think of y as providing additional information that allows you to confirm when $x \in C$, but without the possibility of looping.)

7.* (Optional) Let M be a single tape TM that cannot write on the portion of the tape which contains the input string. Show that $L(M)$ is a regular language.