Problem Set 1

Read all of Chapters 1 and 2 except Section 2.4.

0.1 Read and solve, but do not turn in: Book, 1.14. [swapping NFA accept/non-accept states]

0.2 Read and solve, but do not turn in: Book, 1.31. [closure under reversal]

0.3 Read and solve, but do not turn in: Book, 1.46c. [Pumping lemma]

1. Let \( C_n = \{ x \mid x \text{ is a binary number that is a multiple of } n \} \). Show that, for each \( n \geq 1 \), the language \( C_n \) is regular, by giving the formal description of a DFA \( D_n \) that recognizes \( C_n \).

2. The Hamming distance \( H(x, y) \) between two strings \( x \) and \( y \) of equal length, is the number of corresponding symbols at which \( x \) and \( y \) differ. For example, \( H(1101111, 0001111) = 2 \).

For any language \( A \), let \( N_1(A) = \{ w \mid H(w, x) \leq 1 \text{ for some } x \in A \} \).

Show that the class of regular languages is closed under the \( N_1 \) operation.

3. (a) Let \( B = \{ 1^k y \mid y \in \{0, 1\}^* \text{ and } y \text{ contains at least } k \text{ 1s, for } k \geq 1 \} \).

Show that \( B \) is a regular language.

(b) Let \( C = \{ 1^k y \mid y \in \{0, 1\}^* \text{ and } y \text{ contains at most } k \text{ 1s, for } k \geq 1 \} \).

Show that \( C \) isn’t a regular language.

4. Let \( M_1 \) and \( M_2 \) be DFAs that have \( k_1 \) and \( k_2 \) states, respectively, and let \( U = L(M_1) \cup L(M_2) \).

(a) Show that if \( U \neq \emptyset \), then \( U \) contains some string \( s \), where \( |s| < \max(k_1, k_2) \).

(b) Show that if \( U \neq \Sigma^* \), then \( U \) excludes some string \( s \), where \( |s| < k_1k_2 \).

5. Let \( x \) and \( y \) be strings over some alphabet \( \Sigma \). Say \( x \) is a substring of \( y \) if \( y \in \Sigma^* x \Sigma^* \) and say \( x \) is a major substring of \( y \) if \( x \) is a substring of \( y \) and \( |x| \geq \frac{1}{2}|y| \).

For any language \( B \), let \( MS(B) = \{ x \mid x \text{ is a major substring of } y \text{ for some } y \in B \} \).

Show that if \( B \) is regular then \( MS(B) \) is context-free.

6. Consider the following CFG \( G \):

\[
S \rightarrow aSb \mid aSbb \mid \varepsilon
\]

Describe \( L(G) \) and show that \( G \) is ambiguous.

Give an unambiguous grammar \( H \) where \( L(H) = L(G) \) and prove that \( H \) is unambiguous.

7. (optional) Strengthen Problem 5 by showing that if \( B \) is regular then \( MS(B) \) is also regular.