Problem Set 1

Problem 1. Read and solve, but do not turn in: Book, 1.14. [swapping NFA accept/non-accept states]

Problem 2. Read and solve, but do not turn in: Book, 1.31. [closure under reversal]

Problem 3. Read and solve, but do not turn in: Book, 1.46b. [Pumping lemma]

You can assume the results from the above problems when solving the problems below.

1. Let \( \Sigma_2 = \{ \[0\], \[1\], \[0,1\], \[1,0\] \}. \) Here, \( \Sigma_2 \) contains all columns of 0s and 1s of height two. A string of symbols in \( \Sigma_2 \) gives two rows of 0s and 1s. Consider each row to be a binary number and let

\[
T = \{ w \in \Sigma_2^2 \mid \text{the bottom row of } w \text{ is three times the top row} \}.
\]

For example, \( \[0] \[0] \[1] \[0] \in T \), but \( \[0] \[0] \[0] \[0] \not\in T \). Show that \( T \) is regular.

2. Let \( \Sigma \) be any language over an alphabet \( \Sigma \). Define \( ADD-ONE(A) \) to be the language containing all strings that can be obtained by adding a symbol in \( \Sigma \) anywhere to a string in \( A \). Thus, \( ADD-ONE(A) = \{ xay \mid xy \in A \text{ where } x, y \in \Sigma^*, a \in \Sigma \} \).

Show that the class of regular languages is closed under the \( ADD-ONE \) operation.

3. For any regular expression \( R \) and \( k \geq 0 \), let \( R^k \) be \( R \) self-concatenated \( k \) times, \( \underbrace{RR\cdots R}_{\text{k times}} \).

Let \( \Sigma = \{0,1\} \).

(a) Let \( A = \{ \theta^k u 1^k \mid k \geq 1 \text{ and } u \in \Sigma^* \} \). Show \( A \) is regular.

(b) Let \( B = \{ \theta^k u 1^k \mid k \geq 1 \text{ and } u \in 1\Sigma^* \} \). Show \( B \) is not regular.

4. String \( w \) is a palindrone if \( w = w^R \). Let \( NEP \) be the language of all strings over \( \Sigma = \{0,1\} \) that are not even-length palindromes. Prove that \( NEP \) is not a regular language.

5. Let \( \Sigma = \{0,1\} \). For \( k \geq 1 \), let \( E_k = \{ w \mid |w| \geq k \text{ and the } k \text{th symbol from the end of } w \text{ is a } 1 \} \).

Here, \( |w| \) means the length of \( w \).

(a) Given \( k \), describe a regular expression for \( E_k \). You may use the exponentation notation given in problem 3.

(b) Given \( k \), describe an NFA with \( k+1 \) states for \( E_k \), with a picture and a formal description.

(c) Prove that for each \( k \), no DFA can recognize \( E_k \) with fewer than \( 2^k \) states.

6. (a) Use CFGs to show that the class of CFLs is closed under union.

(b) Let \( E = \{ a^i b^j \mid i \neq j \text{ and } 2i \neq j \} \). Use part (a) to show that \( E \) is a context-free language.

(Hint: Express \( E \) in a different way.)

7. (*) Let \( M = (Q, \Sigma, \delta, q_0, F) \) be a DFA and let \( b \) be a state of \( M \) called its “base”. A reset string for \( M \) and \( b \) is a string \( s \in \Sigma^* \) where \( \delta(q, s) = b \) for every \( q \in Q \). (Here we have extended \( \delta \) to strings, so that \( \delta(q, s) \) equals the state where \( M \) ends up when \( M \) starts at state \( q \) and reads input \( s \).) Say that \( M \) is resettable if it has a reset string for some state \( b \).

Prove that if \( M \) is a \( k \)-state resettable DFA, then it has a reset string of length at most \( k^3 \). (Note: I believe it is unknown whether the bound can be improved to \( o(k^3) \).)