18.404/6.5400 Michael Sipser
Theory of Computation
Due via Gradescope at noon sharp, Thursday, September 21, 2023 **Problem Set 1**

Read all of Chapters 1 and 2 except Section 2.4.

0.1 Read and solve, <u>but do not turn in</u>: Book, 1.14 . [Swapping NFA accept/non-accept states]

0.2 Read and solve, <u>but do not turn in</u>: Book, 1.31 . [Closure under reversal]

0.3 Read and solve, <u>but do not turn in</u>: Book, 1.46b . [Pumping lemma]

0.4 Read and solve, <u>but do not turn in</u>: Book, 2.4 and 2.5 . [Practice with CFLs]

You can assume the results from the above problems when solving the problems below.

1. Let
$$\Sigma_3 = \left\{ \begin{bmatrix} 0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \ldots, \begin{bmatrix} 1\\1\\1 \end{bmatrix} \right\}.$$
Σ_3 contains all size 3 columns of 0s and 1s. A string of symbols in Σ_3 gives three rows of 0s and 1s. Consider each row to be a binary number and let
$$B = \{w \in \Sigma_3^* |\ \text{the bottom row of } w \text{ is the sum of the top two rows}\}.$$
For example, $\begin{bmatrix} 0\\0\\1 \end{bmatrix}\begin{bmatrix} 1\\0\\0 \end{bmatrix}\begin{bmatrix} 1\\1\\0 \end{bmatrix} \in B$, but $\begin{bmatrix} 0\\0\\1 \end{bmatrix}\begin{bmatrix} 1\\0\\1 \end{bmatrix} \notin B$. Show that B is regular.

2. Let Σ_3 be the same as in Problem 1. Let
$$M = \{w \in \Sigma_3^* |\ \text{the bottom row of } w \text{ is the product of the top two rows}\}.$$
For example, $\begin{bmatrix} 0\\0\\1 \end{bmatrix}\begin{bmatrix} 0\\0\\0 \end{bmatrix}\begin{bmatrix} 1\\1\\0 \end{bmatrix}\begin{bmatrix} 1\\1\\1 \end{bmatrix} \in M$, but $\begin{bmatrix} 0\\0\\1 \end{bmatrix}\begin{bmatrix} 1\\1\\1 \end{bmatrix}\begin{bmatrix} 1\\0\\1 \end{bmatrix} \notin M$. Show that M is not regular.

3. Let A be any language. Define $\mathit{DROP\text{-}ONE}(A)$ to be the language containing all strings that can be obtained by removing one symbol from a string in A.
Thus, $\mathit{DROP\text{-}ONE}(A) = \{xz |\ xyz \in A \text{ where } x, z \in \Sigma^*, y \in \Sigma\}$.
Show that the class of regular languages is closed under the $\mathit{DROP\text{-}ONE}$ operation.
Give both a proof by picture and a more formal proof by construction as in Theorem 1.47.

4. Let $\Sigma = \{0, 1\}$. Let $WW_k = \{ww |\ w \in \Sigma^* \text{ and } w \text{ is of length } k\}$. Show that for each k:
   (a) No DFA can recognize WW_k with fewer than 2^k states.
   (b) Some NFA with $O(k)$ states can recognize $\overline{WW_k}$, the complement of WW_k.
   Here $O(k)$ means at most ck for some constant c independent of k.

5. Let $\Sigma = \{0, 1\}$.
   (a) Let $A = \{0^k u 0^k |\ k \geq 1 \text{ and } u \in \Sigma^*\}$. Show A is regular.
   (b) Let $B = \{0^k 1 u 0^k |\ k \geq 1 \text{ and } u \in \Sigma^*\}$. Show B is not regular.

6. Let $\Sigma = \{0, 1\}$ and let $C_1 = \{trt |\ t \in 0^* \text{ and } r \in 0^*10^*, \text{ where } |t| = |r|\}$. The notation $|x|$ means the length of string x. Show that C_1 is a CFL, by giving a CFG *and* by giving a PDA. You do not need to prove that your solutions work, but please give comments to assist the grader. (Hint: This problem is tricky but not complicated. It has a CFG with three rules.)

7.* ($\star$ means optional) For any language A, let $RC(A) = \{yx |\ xy \in A\}$.
Show that the class of regular languages is closed under the RC operation.