Notes 8.370/18.435 Fall 2022

Lecture 14 Prof. Peter Shor

The Nobel Prize in Physics was given out this year to Alain Aspect, John Clauser, and Anton Zeilinger. In honor of that, we are going to explain the GHZ paradox, one of the many things that Anton Zeilinger won the Nobel Prize for on Tuesday. This paradox is named for its discoverers, Greenberger, Horne, and Zeilinger. In the next set of lecture notes, we will describe an experiment that Zeilinger did demonstrating the GHZ paradox.

These lecture notes cover something like half a class period, but I'm writing them up as a single lecture because it's easier that way (they were combined with a quite different subject).

The GHZ paradox can be viewed as a game. It wasn't at first, but computer scientists love putting things in terms of games, because this has been a very fruitful way of thinking about several aspects of computer science. So when computer scientists started looking at quantum paradoxes, this was a natural way to view them.

This game is played by three players, Alice, Bob, and Carol, who are all cooperating, and a referee who plays randomly. Alice, Bob, and Carol cannot communicate once the game starts, but they can meet beforehand and agree on a strategy. The referee gives Alice, Bob, and Carol each a bit, which is either an X or a Y. He always gives an odd number of Xs, so either there are three Xs or one X and two Ys. More specifically, he gives the three players XXX, YYX, YXY, and XYY with equal probabilities.

When the players get these bits, they have to give a bit back to the referee, which will either be a $+$ or a $-$. The referee looks at the bits he receives, and decides whether the players win. The rule is that if all three players get an X, they must return an odd number of $+$s. However, if two players get a Y and one an X, the must return an even number of $+$s.

Can the players win this game with a classical strategy? The answer is "no", they can only win $3/4$ of the time. One way they can do this is to always return an even number of Xs, so for instance they could all return Ys. Then they win unless the referee gives them XXX, which only happens $1/4$ of the time.

Why can't the players do better? Let's consider what happens when they use a deterministic strategy. It's fairly easy to use probability theory to show that a probabilistic strategy cannot do better than a deterministic one, but we won't include the proof in these notes.

So what is a deterministic strategy? It's a table telling what each of Alice, Bob, and Carol will return if the referee gives them an X or a Y. For example, this might be the strategy:

$player$	X	Y
A	$+$	$-$
B	$+$	$-$
C	$+$	$+$

In this strategy, if they get XXX, they return $+ + +$, so they win. If they get XYY they return $+ - +$, so they win. Similarly, if they get YXY, they return $- + +$, so they win.

But if they get YYX, they return $--+$, so they lose.

So let's consider the strategy table.

$player$	X	Y
A	a	d
B	b	e
C	c	f

There are only four possible challenges the referee can give the players, XXX YYX, YXY, XYY. The players' responses to these are abc, dec, dbf, aef, respectively. They must return an odd number of $+$'s in one case, and an even number of $+$'s in the other three. Thus, the set of responses

$$\{abc, dec, dbf, aef\}$$

must contain an odd number of $+$'s altogether. However, this is impossible, since each letter appears exactly twice in the sequence abc, dec, dbf, aef, so no matter how you assign $+$ and $-$ to the letters, the total number of $+$'s must be even.

Now, how can they win with probability 1 by using quantum mechanics? What they do is share what is called a GHZ state before the game. This is the state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Then each of the players measures the state in either the x or the y basis, depending on whether they were given on X or a Y.

Suppose they were given three X's. Then the probability they measure $|+++\rangle$ is

$$\frac{1}{\sqrt{2}^4}(\langle 0| + \langle 1|)(\langle 0| + \langle 1|)(\langle 0| + \langle 1|)(|000\rangle + |111\rangle) = \frac{1}{4}(1+1)$$

You can see that there are only two terms that appear in the final sum: $\langle 0|\langle 0|\langle 0|000\rangle$ and $\langle 1|\langle 1|\langle 1|111\rangle$, and they both contribute $\frac{1}{4}$ to the amplitude. The amplitude is thus $\frac{1}{2}$, and the probability of getting this oucome is its square, $\frac{1}{4}$. Similarly, the probability of getting the outcome $|+--\rangle$ is

$$\frac{1}{\sqrt{2}^4}(\langle 0| + \langle 1|)(\langle 0| - \langle 1|)(\langle 0| - \langle 1|)(|000\rangle + |111\rangle) = \frac{1}{4}(1+1),$$

because the two -1 coefficients from the two $|-\rangle$ multiply to give $+1$. However, if we have just one $-$, as in $|+-+\rangle$, then the amplitude coming from the term $|111\rangle$ would be -1, and the amplitude from the term $|000\rangle$ would still be 1. Adding these gives 0 meaning that the probability of seeing exactly one $-$ is 0.

It is fairly easy to see that if the players were given XYY, then the chance of seeing $+++$ is

$$\frac{1}{\sqrt{2}^4}(\langle 0| + \langle 1|)(\langle 0| + i\langle 1|)(\langle 0| + i\langle 1|)(|000\rangle + |111\rangle) = \frac{1}{4}(1-1),$$

as the two i's multiply to give -1. Thus the probability of seeing $|+++\rangle$ is 0. You need an even number of $+$s in an outcome to get $1+1$, and thus a chance of seeing that outcome.

Thus, if they are allowed to share a GHZ state, the players can measure it and win with probability 1.