Notes 8.370/18.435 Fall 2022

Lecture 11 Prof. Peter Shor

Today, we will be talking about quantum teleportation.

Note: This lecture took a little more than a class period,

What is quantum teleportation? There is a theorem (which we will prove shortly) that you cannot send quantum information over a classical channel. More specifically, you cannot send an unknown quantum state $|\psi\rangle$ from Alice to Bob if the only communication channels connecting them are classical.

However, if Alice and Bob share an EPR pair, Alice can send Bob an unknown quantum state $|\psi\rangle$ using an EPR pair and two bits communicated on a classical channel. This process is known as quantum teleportation, and destroys the EPR pair and Alice's copy of $|\psi\rangle$.

Theorem 1 *If Alice and Bob have a classical channel between them, but do not share any entanglement, then Alice cannot send an unknown quantum state to Bob.*

We want to show that you cannot send an unknown quantum state over a classical channel. (If you have a known quantum state, you can send an arbitrarily precise description of it, just by sending a message like "$0.809017\,|0\rangle + 0.587785\,|1\rangle$".)

We will prove the theorem by contradiction. Let's say that Alice receives an unknown quantum state $|\psi\rangle$, and can encode it in a classical channel, which she then sends to Bob, who can reconstruct $|\psi\rangle$. There is nothing preventing Alice from duplicating the classical information going down the channel, so she could then send the same information to Carol, who could also reconstruct $|\psi\rangle$, using the same recipe Bob uses. We have thus cloned $|\psi\rangle$, a contradiction,so we have proved the theorem.

However, you can send a quantum state over a classical channel if Alice and Bob have an entangled state. More precisely, we will show if they have an EPR pair of qubits in the state $\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$, with Alice holding one of these qubits and Bob the other, then Alice can send Bob the state of an unknown qubit by using two classical bits.

This does not allow us to clone, because while a third party, Carol, could copy the two classical bits sent, she does not share the EPR pair that was used to teleport the qubit. Without the EPR pair, the classical bits used to teleport are completely random, so that she learns nothing from them. Further, the unknown qubit that Alice is sending over the channel is measured during the teleportation operation, so Alice transmits the state of the unknown qubit, but does not clone it. Thus, quantum teleportation doesn't contradict our no-go theorem.

Before I describe teleportation, we need a little background. The *Bell basis* for a two qubit state space is

$$\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big), \quad \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big), \quad \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big), \quad \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big).$$

These form an orthonormal basis for the set of two qubits. Each of these is called a *Bell state*, and each of Alice and Bob can switch from one Bell state to another by applying a Pauli matrix to their qubit:

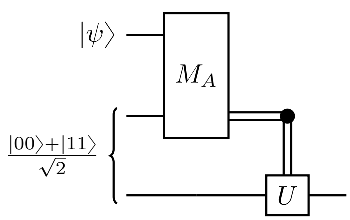
$$\frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) = \big(\sigma_z \otimes I\big)\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)$$
$$\frac{1}{\sqrt{2}}\big(\,|01\rangle + |10\rangle\,\big) = \big(\sigma_x \otimes I\big)\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)$$
$$\frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big) = i\big(\sigma_y \otimes I\big)\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big).$$

How does it work? In this lecture, I will first give the teleportation protocol, and show that it works for one of Alice's measurement results by explicit calculation. I will then use a clever manipulations of formulas to show that it works in the remaining three cases. Finally, i will derive the quantum circuit for teleportation.

So how does teleportation work? Alice holds a qubit in a state she does not know, which we will call $|\psi\rangle = \alpha\,|0\rangle_{A_1} + \beta\,|1\rangle_{A_1}$. There are also two qubits in the state $\frac{1}{\sqrt{2}}\big(\,|00\rangle_{A_2B} + |11\rangle_{A_2B}\,\big)$, of which Alice holds the first and Bob the second. Thus, the state is

$$\frac{1}{\sqrt{2}}\big(\alpha\,|0\rangle_{A_1} + \beta\,|1\rangle_{A_1}\,\big)\big(\,|00\rangle_{A_2B} + |11\rangle_{A_2B}\,\big)$$

What is the teleportation protocol? Alice measures the first two qubits of her state using the Bell basis, and then sends the results of the measurement (which can be encoded in two bits, since there are four outcomes) to Bob. Bob then applies a unitary to his state depending on the measurement. So in broad outline, this looks like:



One way to show that this works is do four computations, one for each of the possible measurement results that Alice gets. This isn't hard, but you'll learn more from my showing you a cleverer way. This way involves doing the computation explicitly for one of the four measurement outcomes, and deriving the other three from this one.

First, let's do the case where Alice gets $\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)$ as the result of her measurement:

$$\tfrac{1}{2}\big({}_{A_1A_2}\langle 00| + {}_{A_1A_2}\langle 11|\,\big)\big(\alpha\,|0\rangle_{A_1} + \beta\,|1\rangle_{A_1}\,\big)\big(\,|00\rangle_{A_2B} + |11\rangle_{A_2B}\,\big) = \tfrac{1}{2}\big(\alpha\,|0\rangle_B + \beta\,|1\rangle_B\,\big)$$

The ${}_{A_1A_2}\langle 00|$ picks out the term $\alpha\,|000\rangle_{A_1A_2B}$ and, after the inner product, leaves $\alpha\,|0\rangle_B$. Similarly, the term ${}_{A_1A_2}\langle 11|$ picks out the term $\beta\,|111\rangle_{A_1A_2B}$ and leaves $\beta\,|1\rangle_B$. Thus, the measurement outcome occurs with probability $\big(\frac{1}{2}\big)^2 = \frac{1}{4}$, and Bob gets Alice's original unknown state $\alpha\,|0\rangle + \beta\,|1\rangle = |\psi\rangle$.

For our next calculation, we need some identities on the Bell states:

$$
\begin{aligned}
\Big(\sigma_x \otimes I\Big)\frac{|00\rangle + |11\rangle}{\sqrt{2}} &= \quad \Big(I \otimes \sigma_x\Big)\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|19\rangle + |01\rangle}{\sqrt{2}} \\
\Big(\sigma_z \otimes I\Big)\frac{|00\rangle + |11\rangle}{\sqrt{2}} &= \quad \Big(I \otimes \sigma_z\Big)\frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
\Big(\sigma_y \otimes I\Big)\frac{|00\rangle + |11\rangle}{\sqrt{2}} &= -\Big(I \otimes \sigma_y\Big)\frac{|00\rangle + |11\rangle}{\sqrt{2}} = i\frac{|10\rangle - |01\rangle}{\sqrt{2}}
\end{aligned}
$$

These properties are fairly easy to prove, so I won't do it in these notes.

Suppose Alice gets the result $\frac{1}{\sqrt{2}}(I \otimes \sigma_w)(|00\rangle + |11\rangle)$ where $w = x$ or z. Then we have that the state of the system is

$$\tfrac{1}{2}\big({}_{A_1A_2}\langle 00| + {}_{A_1A_2}\langle 11|\big)\big(I_{A_1} \otimes \sigma_{w(A_2)}\big)\big(\alpha\,|0\rangle_{A_1} + \beta\,|1\rangle_{A_1}\big)\big(\,|00\rangle_{A_2B} + |11\rangle_{A_2B}\big)$$

But now, $\sigma_{w(A_2)}$ doesn't affect qubit A_1, so we can move this Pauli and get

$$\tfrac{1}{2}\big({}_{A_1A_2}\langle 00| + {}_{A_1A_2}\langle 11|\big)\big(\alpha\,|0\rangle_{A_1} + \beta\,|1\rangle_{A_1}\big)\big(\sigma_{w(A_2)} \otimes I_B\big)\big(\,|00\rangle_{A_2B} + |11\rangle_{A_2B}\big)$$

Now, by the above identities on the Bell basis, this is the same as

$$\tfrac{1}{2}\big({}_{A_1A_2}\langle 00| + {}_{A_1A_2}\langle 11|\big)\big(\alpha\,|0\rangle_{A_1} + \beta\,|1\rangle_{A_1}\big)\big(I_{A_2} \otimes \sigma_{w(B)}\big)\big(\,|00\rangle_{A_2B} + |11\rangle_{A_2B}\big)$$

(this is where we use $w = x$ or z). But because $\sigma_{w(B)}$ only affects Bob's qubit, we can move it all the way to the left, across Alice's qubits and operations, to get

$$\tfrac{1}{2}\sigma_{w(B)}\big({}_{A_1A_2}\langle 00| + {}_{A_1A_2}\langle 11|\big)\big(\alpha\,|0\rangle_{A_1} + \beta\,|1\rangle_{A_1}\big)\big(\,|00\rangle_{A_2B} + |11\rangle_{A_2B}\big)$$

However, except for the $\sigma_{w(B)}$, this is exactly what we get when Alice measures her first outcome, and we've computed this before. So we are left with

$$\sigma_{w(B)}\big(\alpha\,|0\rangle_B + \beta\,|1\rangle_B\big) = \sigma_w\,|\psi\rangle\,,$$

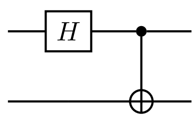
and if Bob applies σ_w (he knows what w is because Alice tells him the result of her measurement), he gets $|\psi\rangle$, the state Alice wanted to teleport. The third case, when Alice measures $\frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\big)$, works exactly the same way except for the phases, which don't matter because they are global phases.

How can we figure out a quantum circuit for teleportation? There are two elements to this: first, we need to figure out how to measure in the Bell basis, and second, we need how to apply the correct Pauli matrix to correct the state to $|\psi\rangle$.

How do we figure out how to measure in the Bell basis? The easiest way to do it is to work backwards. We want to find a circuit where you input one of the elements of the Bell basis, and where we output 00, 01, 10, 11. after a measurement. Let's start by figuring out a circuit where we input $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, and output a member of the Bell basis. This is actually fairly easy to do. First, we need to make a superposition of states at some point, and then we need to entangle the two qubits. To make a superposition of states, we use the Hadamard gate, $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and to entangle the qubits we use the CNOT gate:

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So the quantum circuit we will try is:

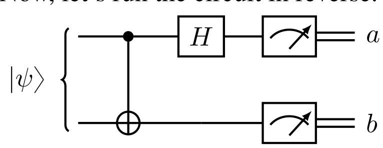


What happens with this circuit? Here is a table of the inputs, the intermediate states, and the outputs:

input	after H	output
$\|00\rangle$	$\frac{1}{\sqrt{2}}\big(\|0\rangle + \|1\rangle\big)\|0\rangle$	$\frac{1}{\sqrt{2}}\big(\|00\rangle + \|11\rangle\big)$
$\|01\rangle$	$\frac{1}{\sqrt{2}}\big(\|0\rangle + \|1\rangle\big)\|1\rangle$	$\frac{1}{\sqrt{2}}\big(\|01\rangle + \|10\rangle\big)$
$\|10\rangle$	$\frac{1}{\sqrt{2}}\big(\|0\rangle - \|1\rangle\big)\|0\rangle$	$\frac{1}{\sqrt{2}}\big(\|00\rangle - \|11\rangle\big)$
$\|11\rangle$	$\frac{1}{\sqrt{2}}\big(\|0\rangle - \|1\rangle\big)\|1\rangle$	$\frac{1}{\sqrt{2}}\big(\|01\rangle - \|10\rangle\big)$

So this does what we want it to do.

Now, let's run the circuit in reverse.

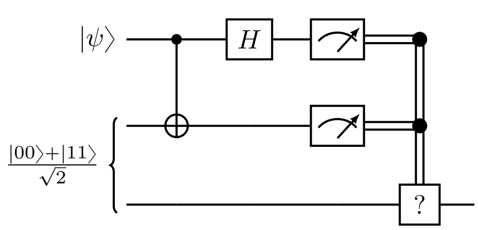


The probability of getting output a, b is

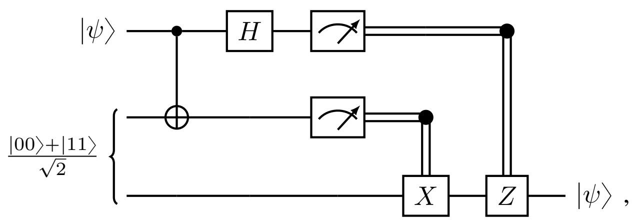
$$\left|\langle ab|\, H_1 \mathrm{CNOT}_{1\to 2}\, |\psi\rangle\right|^2,$$

so this measures the input state in the basis $\langle ab|\, H_1\mathrm{CNOT}_{1\to 2}$, where a and b are either 0 or 1, which gives a measurement in the Bell basis.

So now, our quantum circuit looks like:



And all we need to do is figure out which unitary we need to use to make the correction. Recall that we showed that we need to make the correction σ_w when Alice measures the state $(\sigma_w \otimes I)\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$. By the table above, if the first measurement result is $|1\rangle$, we need to apply a σ_z and if the second measurement bit is $|1\rangle$, we need to apply a σ_x. This gives the circuit



so we have derived a quantum circuit for teleportation.

We can represent quantum teleportation schematically as follows:

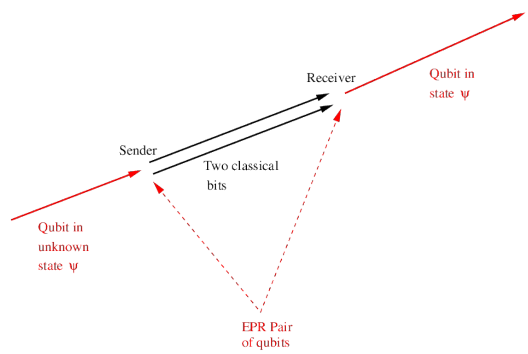


Figure 1: teleportation

Time goes up in this figure, so the first thing that happens is that the sender and receiver share an EPR pair. Then, the sender encodes her unknown qubit and sends it to the receiver, who decodes it with the help of his half of the EPR pair.

There is a converse process to teleortation: superdense coding. Here, if they share an EPR pair, the sender and receiver can send two classical bits using one quantum bit. A schematic representation of this process is:

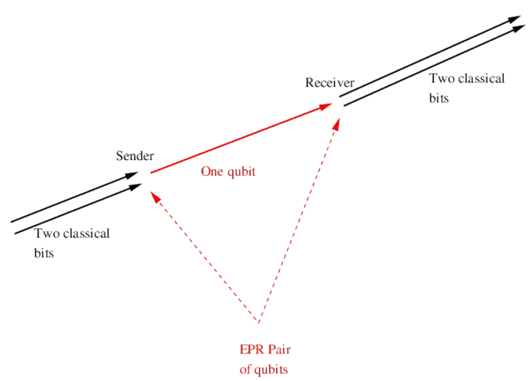


Figure 2: superdense coding

As we will see, for superdense coding, the sender encodes using the same process the receiver uses to decode in teleportation, and the receiver decodes using the process the sender uses to encode in teleportation.

How does the process work? Recall the Bell basis is four entangled states, and can be obtained from the state $|00\rangle + |11\rangle$ by applying a Pauli matrix. We have:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big) \\
&\frac{1}{\sqrt{2}}\big(\,|00\rangle - |11\rangle\,\big) = \frac{1}{\sqrt{2}}(\sigma_z \otimes I)\big(\,|00\rangle + |11\rangle\,\big) \\
&\frac{1}{\sqrt{2}}\big(\,|01\rangle + |10\rangle\,\big) = \frac{1}{\sqrt{2}}(\sigma_x \otimes I)\big(\,|00\rangle + |11\rangle\,\big) \\
&\frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big) = \frac{1}{\sqrt{2}}(\sigma_z \sigma_x \otimes I)\big(\,|00\rangle + |11\rangle\,\big)
\end{aligned}$$

Thus, Alice can convert the EPR pair she shares with Bob to any one of the Bell basis states. When she sends her qubit to Bob, he measures in the Bell basis, and gets one of four values, giving him two classical bits. So superdense coding works.

You can also use superdense coding to show that teleportation is optimal. Even with an arbitrarily large number of Bell pairs, you cannot send a qubit using fewer than two classical bits. Why not? Suppose you could find a protocol that sent a qubit using 1.9 classical bits on average. Then, encoding two classical bits using superdense coding, and encoding the resulting qubit with the improved teleportation protocol, you could send 2 classical bits using entanglement and 1.9 classical bits on average. Repeating this k times, for large n you could send n classical bits using $(0.95)^k n$ classical bits on average. While we won't prove it in class, Shannon's channel capacity theorem shows that this in turn would let you send classical information faster than the speed of light using just entanglement, which we assume is impossible from Einstein's theory of relativity.