Notes 8.370/18.435 Fall 2022

Lecture 6 Prof. Peter Shor

In the last class, you saw how the tensor product of quantum states and the tensor product of unitary operators worked. Today, we'll talk about the tensor product of measurements.

Recall that a von Neumann measurement on a quantum state space A was a set of Hermitian projectors $\Pi_1, \Pi_2, \ldots, \Pi_k$ such that $\Pi_i \Pi_j = 0$ if $i \neq j$ and $\sum_{i=1}^{k} \Pi_i = I_A$.

Suppose you have two individual von Neumann measurements on systems A and B, $\{\Pi_1^A, \Pi_2^A, \ldots, \Pi_k^A\}$ and $\{\Pi_1^B, \Pi_2^B, \ldots, \Pi_\ell^B\}$. Then there is a tensor product measurement on $A \otimes B$ given by

$$\{\Pi_i^A \otimes \Pi_j^B | 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \ell\}.$$

It is not difficult to show that this set of projectors satisfies the conditions to be a von Neumann measurement.

One very common tensor product measurement is when we just make a measurement on system A and leave system B unchanged. This operation can be expressed in the above formulation by letting the measurement on B be the single projector I_B (which does nothing to the quantum state on B). Thus if we have a basis for a qubit $|v\rangle$, $|\bar{v}\rangle$ on systeam A, measurement operators for system AB are $|v\rangle_A \langle v| \otimes I_B$ and $|\bar{v}\rangle_A \langle \bar{v}| \otimes I_B$.

Let's do an example. Suppose we have two qubits in the joint state

$$|\psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle$$

and we measure the first qubit in the $\{0, 1\}$ basis. What happens?

The projection matrices associated with this measurement are

$$|0\rangle\langle 0| \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad |1\rangle\langle 1| \otimes I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What happens? The probability of getting the first outcome is

$$\big|\big(|0\rangle\langle 0| \otimes I\big)|\psi\rangle\big|^2 = \left| \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} \right|^2 = |\alpha|^2 + |\beta|^2$$

and the resulting state is $\frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}(\alpha|00\rangle + \beta|01\rangle)$. Since we measured the first qubit, we know the state of the first qubit, so sometimes (by an abuse of notation or something) we just take the resulting state to be the state on the second qubit, $\frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}(\alpha|0\rangle + \beta|1\rangle)$. Similarly, the probability of getting the second outcome is $|\gamma|^2$, and the resuting state of the second qubit is $|0\rangle$.

There is a different way of doing these calculations. Here, we need to be careful about which qubits the kets and bras apply to, so I will write them explicitly in the following.

We can think of measuring the first qubit as applying either the bra (row vector) $\langle 0\,|$ or $\langle 1\,|$ to the first qubit. We will make this explicit by labeling the qubits A and B. Thus, when we measure the first qubit, we get one of the following two (unnormalized) states:

$$\begin{aligned}
{}_A\langle 0\,|\,\big(\alpha\,|\,00\rangle_{AB} + \beta\,|\,01\rangle_{AB} + \gamma\,|\,00\rangle_{AB}\big) &= \alpha\,|\,0\rangle_B + \beta\,|\,1\rangle_B \\
{}_A\langle 1\,|\,\big(\alpha\,|\,00\rangle_{AB} + \beta\,|\,01\rangle_{AB} + \gamma\,|\,00\rangle_{AB}\big) &= \gamma\,|\,0\rangle_B
\end{aligned}$$

How do these calculations work? We have ${}_A\langle 0|00\rangle_{AB} = |\,0\rangle_B$ and ${}_A\langle 0|10\rangle_{AB} = 0$. So applying ${}_A\langle\cdot\,|$ to $|\,\cdot\rangle_{AB}$ results in an unnormalized quantum state vector on qubit B. The square of length of this vector gives the probability of that measurement outcome, and the normalized vector gives the resulting quantum state.

We now talk about the tensor product of observables. Suppose we have two observables M_A on quantum system A and M_B on quantum system B. Recall that if we have an observable M applied to a quantum state $|\,\psi\rangle$, it returns a value, and that $\langle\psi\,|\,M\,|\,\psi\rangle$ is the expectation of the value.

How do we get the observable for the product of the values? We use $M_A \otimes M_B$. How about for the sum of the values? We use $M_A \otimes I_B + I_A \otimes M_B$, where I_A and I_B are the identity matrices on systems A and B, respectively.

Let's take an example. The angular momentum observable for a spin-$\frac{1}{2}$ particle is $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}$. Suppose we have two particles, A and B. Then

$$J_A \otimes J_B = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$J_A \otimes I_b + I_A \otimes J_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now, let's look at the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$. I claim that if you measure both qubits in the same basis, no matter which basis it is, the two qubits will be different, i.e., they will be orthogonal. What this means physically is that if you have the state $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$, and you measure them along any axis, you will find that they are always spinning in opposite directions; a physical way of understanding this is that the total spin of the state $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$ is 0, so when you meaaure each qubit, you end up with a total spin of 0.

Let's do the computation. Suppose we measure the first particle in the basis

$$\{\cos(\theta)\,|\,0\rangle + e^{i\phi}\sin(\theta)\,|\,1\rangle\,, \sin(\theta)\,|\,0\rangle + e^{-i\phi}\cos(\theta)\,|\,1\rangle\}.$$

(It is straightforward to check that these two states are orthogonal and thus form a basis.) Then

$$\frac{1}{\sqrt{2}}\Big(\cos(\theta)\,\langle 0\,| + e^{-i\phi}\sin(\theta)\,\langle 1\,|\Big)(|\,01\rangle - |\,10\rangle) = \frac{1}{\sqrt{2}}\cos(\theta)\,|\,1\rangle - e^{-i\phi}\sin(\theta)\,|\,0\rangle\,.$$

This is indeed the orthogonal state, so we have that if the two qubits are measured along the same axis, the two states are orthogonal.

Now, we know enough to explain a variation of the EPR thought experiment. Suppose Alice and Bob have an entangled pair of qubits in the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$ (called an *EPR pair*). They can separate them far enough so that they measure them simultaneously, so that the speed of light prevents Alice's particle from communicating the basis its being measured in to Bob's particle.

Now, Alice and Bob measure their particle in one of the four following bases:

$$\begin{array}{lll}
\{ \quad |\,f_1\rangle = |\,0\rangle\,, & |\,f_2\rangle = |\,1\rangle & \} \\
\{ \quad |\,g_1\rangle = \cos\frac{\pi}{8}\,|\,0\rangle + \sin\frac{\pi}{8}\,|\,1\rangle\,, & |\,g_2\rangle = -\sin\frac{\pi}{8}\,|\,0\rangle + \cos\frac{\pi}{8}\,|\,1\rangle & \} \\
\{ \quad |\,h_1\rangle = \cos\frac{\pi}{4}\,|\,0\rangle + \sin\frac{\pi}{4}\,|\,1\rangle\,, & |\,h_2\rangle = -\sin\frac{\pi}{4}\,|\,0\rangle + \cos\frac{\pi}{4}\,|\,1\rangle & \} \\
\{ \quad |\,j_1\rangle = \cos\frac{3\pi}{8}\,|\,0\rangle + \sin\frac{3\pi}{8}\,|\,1\rangle\,, & |\,j_2\rangle = -\sin\frac{3\pi}{8}\,|\,0\rangle + \cos\frac{3\pi}{8}\,|\,1\rangle & \}
\end{array}$$

Let's assume that each of Alice and Bob's EPR pairs has a table of outcomes that it will yield for each particle for all these three measurements. We are assuming a deterministic process, but you can show that the same argument applies to probilistic processes. The table will have to look something like this.

Basis F		Basis G		Basis H		Basis J	
A	B	A	B	A	B	A	B
f_1	f_2	g_1	g_2	h_2	h_1	j_2	j_1
f_1	f_2	g_1	g_2	h_1	h_2	j_1	j_2
f_2	f_1	g_2	g_1	h_2	h_1	j_2	j_1
f_2	f_1	g_2	g_1	h_2	h_1	j_1	j_2
f_1	f_2	g_2	g_1	h_2	h_1	j_2	j_1
f_1	f_2	g_1	g_2	h_1	h_2	j_1	j_2

When Alice and Bob measure in the same basis, they must always get opposite outcomes. If Alice measures f_1 in basis F, then Bob must always measure f_2 in basis F. Thus, we can assume that Bob's state is f_2. Now, suppose Bob measures in basis G, then because $\langle f_2|g_2\rangle^2 = \cos^2\frac{\pi}{8} \approx 0.85$, Bob must get g_2 with probability 0.85, so the first two columns must match (i.e., have (f_1, f_2, g_1, g_2) or (f_2, f_1, g_2, g_1)) in around 0.85 of their entries. Similarly, the second and third columns must match in around 0.85 of their entries, and the third and fourth columns must match in around 0.85 of their entries. This means that the first and fourth columns must match in at least $1 - 3 * 0.15 \approx 0.55$ of their entries. However, $\langle f_1|j_2\rangle^2 = \sin^2\frac{\pi}{8} \approx 0.15$, meaning that the first and fourth columns can only match in 0.15 of their entries, a contradiction.

So what does this mean for physical reality? Physicists (and philosophers) have been arguing about this for decades. Is the universe non-local (so the decision as to which basis Alice measures in transmitted faster than light to Bob), or is something else going on? I'm going to leave that question to philosophers. One thing that is true is that even if some information is transmitted faster than light, it's not very useful information. You cannot use entanglement to transmit a message faster than light (although you can use it to do some non-intuitive things, as we will see later in the course).