In the last few lectures, we saw how to operate on one qubit—or more generally, one $d$-dimensional quantum system. We studied how to transform quantum systems with a unitary matrix, and how to measure them via a complete set of projection matrices. In this lecture, we explain how the state space of a joint quantum system, composed of two individual quantum systems, is constructed.

Suppose we have two quantum systems. Each of these has a state space which is a complex vector space of dimensions $d_1$ and $d_2$, respectively. When we consider these two quantum systems together, we get a state space which is the tensor product of their individual state spaces, and which has dimension $d_1 d_2$.

Some of you probably haven’t seen tensor products before. We will show how it works by example. If you have two qubits, each of which has a state space with basis $\{|0\rangle, |1\rangle\}$, then the state space of the joint system has basis $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$.

The convention is to write the basis in lexicographical order. Thus, if you have two qubits in state $\alpha |0\rangle + \beta |1\rangle$ and $\gamma |0\rangle + \delta |1\rangle$, the system of both qubits is in state

$$\alpha_0 \beta_0 |0\rangle \otimes |0\rangle + \alpha_0 \beta_1 |0\rangle \otimes |1\rangle + \alpha_1 \beta_0 |1\rangle \otimes |0\rangle + \alpha_1 \beta_1 |1\rangle \otimes |1\rangle,$$

so the standard distributed law applies to tensor products. Often, rather than writing $|0\rangle \otimes |1\rangle$, we will write this as $|01\rangle$.

To illustrate tensor products using a more usual vector notation,

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \beta_0 \\ \alpha_0 \beta_1 \\ \alpha_1 \beta_0 \\ \alpha_1 \beta_1 \end{pmatrix}.$$

What if you have $n$ qubits? The state space has dimension $2^n$, and has basis $|00\ldots00\rangle, |00\ldots01\rangle, |00\ldots10\rangle, \ldots$. (Recall that by notation, $|00\ldots00\rangle = |0\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \otimes |0\rangle$.)

Now let us count up (real) degrees of freedom. A qubit has a state $\alpha |0\rangle + \beta |1\rangle$, and since $\alpha$ and $\beta$ are complex numbers, this would give four real degrees of freedom. However, it also satisfies $\alpha^2 + \beta^2 = 1$, which means that it only has three real degrees of freedom (if we use the fact that multiplying by a global phase leaves the state essentially unchanged, there are only two degrees of freedom). However, the joint state of two qubits has four complex coefficients, leaving 7 (or 6) degrees of freedom. Thus, because $7 > 2 \cdot 3$, there are some states of the joint system which cannot be the tensor product of states of the individual systems. These states are called entangled. In fact, because the number of degrees of freedom of tensor product states is less than that of entangled states, the tensor product states form a lower-dimensional manifold in the entangled states, and we see that most quantum states are entangled.
For example, the state 
\[
\frac{1}{\sqrt{3}} |00\rangle + \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{6}} |10\rangle + \frac{1}{\sqrt{6}} |11\rangle = \bigg( \frac{\sqrt{2}}{\sqrt{3}} |0\rangle + \frac{1}{\sqrt{3}} |1\rangle \bigg) \bigg( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \bigg)
\]
is a tensor product state, while the state 
\[
\frac{1}{\sqrt{3}} |00\rangle + \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{3}} |10\rangle
\]
is entangled. You can see that the second one is not a tensor product state, and thus entangled, because if we could represent it as a tensor product state 
\[
\alpha_0 \beta_0 |00\rangle + \alpha_0 \beta_1 |01\rangle + \alpha_1 \beta_0 |10\rangle + \alpha_1 \beta_1 |11\rangle,
\]
then we would need \( \alpha_1 \beta_1 = 0 \). But if that holds, either \( \alpha_1 \) or \( \beta_1 \) is 0, so one of \( \alpha_0 \beta_1 \) and \( \alpha_1 \beta_0 \) must be 0, which is not the case.

What about unitary transformations on a joint state space. Suppose we have two unitary matrices \( U \) and \( V \), of dimensions \( k \) and \( \ell \). The tensor product of them is the \( k\ell \times k\ell \) matrix
\[
U \otimes V = \begin{pmatrix}
    u_{11}V & u_{12}V & \cdots & u_{1k}V \\
u_{21}V & u_{22}V & \cdots & u_{2k}V \\
    \vdots & \vdots & \ddots & \vdots \\
u_{k1}V & u_{k2}V & \cdots & u_{kk}V
\end{pmatrix},
\]
where \( u_{ij}V \) is the \((i, j)\) entry of \( U \) multiplied by \( V \).

Let's do an example and figure out what \( \sigma_x \otimes \sigma_z \) is. We have
\[
\sigma_x \otimes \sigma_z = \begin{pmatrix}
    0 & 1 \\
    1 & 0
\end{pmatrix} \otimes \begin{pmatrix}
    1 & 0 \\
    0 & -1
\end{pmatrix} = \begin{pmatrix}
    0 & 0 & 1 & 0 \\
    0 & 0 & 1 & -1 \\
    1 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0
\end{pmatrix}
\]

If the input is in a tensor product state \( |\phi_1\rangle \otimes |\phi_2\rangle \), and you apply a tensor product unitary \( U_1 \otimes U_2 \), then the output is also in a tensor product state, namely \( U_1 |\phi_1\rangle \otimes U_2 |\phi_2\rangle \). So to get an entangled output from a tensor product input, you need to apply a unitary that is a tensor product. One non-tensor-product unitary that we will be using extensively in this class is the CNOT.

We have
\[
CNOT = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 1 & 0
\end{pmatrix} = |0\rangle \otimes \mathbb{I} + |1\rangle \otimes \sigma_x
\]
The last description says that if the first qubit is a \( |0\rangle \), then we apply an identity to the second qubit, and if the first qubit is a \( 1 \), we apply a \( \sigma_x \) (or NOT gate) to the second qubit. This is why it’s called a controlled Not — depending on the state of the first qubit, we apply a NOT gate to the second one (or we don’t).

Note that we don’t measure the first qubit — we don’t get a classical record of the state of the first qubit, and a state that was in a superposition of the first qubit being \( |0\rangle \) and \( |1\rangle \) remains in a superposition. Let’s do an example. Suppose we start with the state \( |+\rangle |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \). When we apply the CNOT gate, the \( |00\rangle \) remains unchanged, because the first qubit is \( |0\rangle \), but the \( |10\rangle \) becomes \( |11\rangle \). This means

\[
\text{CNOT} |+\rangle |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),
\]

so we have used the CNOT gate to entangle two qubits.