Notes 8.370/18.435 Fall 2022

Lecture 4 Prof. Peter Shor

Last time, we started talking about quantum mechanics. We mentioned the principle that *Isolated quantum systems evolve unitarily.* That is, for an isolated system, there is some unitary matrix U_t that takes the state of the system at time $-$, $|\,\phi(0)\rangle$ to the state of the system at time t, ie. $|\,\psi(t)\rangle = U_t\,|\,\Psi(0)\rangle$.

There is another operation we need to know for quantum computation, *measurement*. We have already seem an example of a von Neumann measurement (also called a *projective measurement*). Recall that we said if we have a quantum state $\alpha\,|\,A\rangle + \beta\,|\,B\rangle$, where the $|\,A\rangle$ and $|\,B\rangle$ are completely distinguishable states, then if we apply the experiment that distinguishes $|\,A\rangle$ from $|\,B\rangle$, then we see $|\,A\rangle$ with probability $|\alpha|^2$ and $|\,B\rangle$ with probability $|\beta|^2$. This is probably the simplest example of the following definition:

Definition:

A *complete projective measurement* corresponds to a basis of the system. Suppose $|\,v_1\rangle, |\,v_2\rangle, \ldots |\,v_d\rangle$ is an orthonormal basis for a quantum system. Then, if the system is in state $|\,\psi\rangle$, the von Neumann measurement corresponding to this basis yields $|\,v_i\rangle$ with probability $|\langle v_i|\psi\rangle|^2$, where $\langle v|w\rangle$ is the inner product of the row vector $\langle v\,|$ and the column vector $|\,w\rangle$. Remember also that when you go from $|\,v\rangle$ to $\langle v\,|$, you take the complex conjugate of v, so for a unit vector $|\,v\rangle$, $\langle v|v\rangle = 1$. this is called a *projective measurement* because the quantum state $|\,\psi\rangle$ is projected onto one of the basis vectors $|\,v_i\rangle$: theoretically, after the measurement, the quantum state is $|\,v_i\rangle$. (In practice, sometimes the measurement destroys or alters the quantum state.)

We now show that the probabilities of all the outcomes of a measurement sum to 1. The sum of the probabilities of the outcomes is

$$\sum_i |\langle v_i|\psi\rangle|^2 = \sum_i \langle\psi|v_i\rangle\langle v_i|\psi\rangle.$$

This equality follows from the fact that $\langle\psi|v_i\rangle = \langle v_i|\psi\rangle^*$, so when you multiply them together, you get the real value $|\langle v_i|\psi\rangle|^2$. Now,

$$\sum_i \langle\psi|v_i\rangle\langle v_i|\psi\rangle = \langle\psi\,|\left(\sum_i |\,v_i\rangle\langle v_i\,|\right)|\,\psi\rangle$$

because we can move the sum inside the bracket. Now, since $\{|\,v_i\rangle\}$ forms an orthonormal basis, $\sum_i |\,v_i\rangle\langle v_i\,| = I$ (you will prove this on the homework). So we have

$$\langle\psi\,|\left(\sum_i |\,v_i\rangle\langle v_i\,|\right)|\,\psi\rangle = \langle\psi|\psi\rangle = 1.$$

Maybe I should say a little more about $|\,v\rangle\langle v\,|$. This is just the column vector $|\,v\rangle$ multiplied by the row vector $\langle v\,|$. So if $|\,v\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$|\,v\rangle\langle v\,| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let's now give an example. Suppose we have the quantum state

$$| \psi \rangle = \frac{4}{5} | 0 \rangle + \frac{3}{5} | 1 \rangle .$$

What is the probability of observing each of the outcomes if we measure it in the basis $\{| +\rangle, | -\rangle\}$?

The probability of seeing $| +\rangle$ is

$$\begin{aligned} |\langle + | \psi \rangle|^2 &= \left| \tfrac{1}{\sqrt{2}} (\langle 0 | + \langle 1 |) (\tfrac{4}{5} | 0 \rangle + \tfrac{3}{5} | 1 \rangle) \right|^2 \\ &= \left| \tfrac{4}{5\sqrt{2}} + \tfrac{3}{5\sqrt{2}} \right|^2 = \tfrac{49}{50} \end{aligned}$$

and similarly, the probability of seeing $| -\rangle$ is

$$\begin{aligned} |\langle - | \psi \rangle|^2 &= \left| \tfrac{1}{\sqrt{2}} (\langle 0 | - \langle 1 |) (\tfrac{4}{5} | 0 \rangle - \tfrac{3}{5} | 1 \rangle) \right|^2 \\ &= \left| \tfrac{4}{5\sqrt{2}} - \tfrac{3}{5\sqrt{2}} \right|^2 = \tfrac{1}{50}, \end{aligned}$$

So the chance of the result $| +\rangle$ is $\frac{49}{50}$ while the chance of the result $| -\rangle$ is $\frac{1}{50}$.

This shouldn't be surprising; the vector $(0.8, 0.6)$ is nearly parallel with the vector $| +\rangle = \frac{1}{\sqrt{2}}(1, 1)$, so when you project $| \psi \rangle$ onto $| +\rangle$, you get a large projection, while for $| -\rangle$, you get a small projection.

There is a more general kind of projective measurement, where the outcomes correspond to subspaces rather than quantum states. To specify this kind of measurement, you need to give a set of subspaces $S_1, S_2, \ldots, S_k$. Let Π_i be the projection matrix onto the ith subspace. In other words, if S_i has $v_i^{(1)}, v_i^{(2)} \ldots v_i^{(\ell_i)}$ as an orthogonal basis, then

$$\Pi_i = \sum_{j=1}^{\ell_j} \left| v_i^{(j)} \middle\rangle \middle\langle v_i^{(j)} \right| .$$

This set of subspaces must satisfy the conditions:

$$\Pi_i \Pi_j = 0 \quad \text{if} \quad i \neq j,$$

i.e., the subspaces must be orthogonal, and

$$\sum_i \Pi_i = I,$$

i.e., the subspaces must span the entire quantum state space.

When we apply this measurement to a quantum state $| \psi \rangle$, the probability of seeing the i'th outcome is

$$P(i) = \langle \psi | \Pi_i | \psi \rangle ,$$

and the state of the system after the measurement is

$$\frac{\Pi_i | \psi \rangle}{\langle \psi | \Pi_i | \psi \rangle^{1/2}} ,$$

where we have normalized the projection onto S_i so as to make it a unit vector.

We now give an example. Let's take a four-dimensional quantum system with basis states $|\,0\rangle$, $|\,1\rangle$, $|\,2\rangle$ and $|\,3\rangle$. We want the measurement that asks the question: is the state $|\,0\rangle$ or $|\,1\rangle$, as opposed to being $|\,2\rangle$ or $|\,3\rangle$. That is, we want to the subspace corresponding to the projection matirx $\Pi_A = |\,0\rangle\langle 0\,| + |\,1\rangle\langle 1\,|$ or the projection matrix $\Pi_B = |\,2\rangle\langle 2\,| + |\,3\rangle\langle 3\,|$.

We have

$$\Pi_A + \Pi_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

and

$$\Pi_A \Pi_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 0,$$

so these projection matrices satisfy the conditions needed for them to be a projective measurement.

Let's measure the state $|\,\psi\rangle = \frac{5}{10}\,|\,0\rangle + \frac{7}{10}\,|\,1\rangle + \frac{5}{10}\,|\,2\rangle + \frac{1}{10}\,|\,3\rangle$ with this measurement.

We have

$$\Pi_A\,|\,\psi\rangle = \tfrac{5}{10}\,|\,0\rangle + \tfrac{7}{10}\,|\,1\rangle$$

This gives a probability of $.5^2 + .7^2 = .74$ for this outcome, and after this outcome, the state

$$\frac{1}{\sqrt{.74}}\left(\tfrac{5}{10}\,|\,0\rangle + \tfrac{7}{10}\,|\,1\rangle\right)$$

Similarly,

$$\Pi_B\,|\,\psi\rangle = \tfrac{5}{10}\,|\,2\rangle + \tfrac{1}{10}\,|\,3\rangle.$$

This gives a probability of $.5^2 + .1^2 = .26$ for this outcome, and after this outcome, the state

$$\frac{1}{\sqrt{.26}}\left(\tfrac{5}{10}\,|\,2\rangle + \tfrac{1}{10}\,|\,3\rangle\right)$$

I should probably tell you that there are types of measurements that are more general than these projective measurements, called POVM measurements. We will not discuss them in this class (except maybe in a homework problem or two), because we actually won't need them in this course. You can learn about them by reading the textbook, or taking the follow-up course 8.371/18.436.

There is another way of describing mesurement in quantum mechanics, which is to use *observables*.

Recall from linear algebra that a symmetric matrix M is one for which $M^{\mathrm{T}} = M$, where M^{T} is the transpose of M. A *Hermitian* matrix is one for which $M^{\dagger} = M$, where $M^{\dagger}$ is the Hermitian transpose of M. The *The Hermitian transpose* is the conjugate transpose: you take the transpose of a matrix and you take the complex conjugates of each of its elements.

You can aways diagonalize a Hermitian matrix; if M is Hermitian, then there is a unitary matrix U such that $U^\dagger M U = D$, where D is a diagonal matrix with real entries.

An *observable* is a Hermitian matrix M on the state space of the quantum system. Each of the eigenvalues of M will be associated with a subspace of eigenvectors with that eigenvalue. The subspaces are orthogonal, and span the state space, and thus can be associated with a measurement. For this observable, when the eigenspace associated with the eigenvaue λ_i of M is observed, we say that we have observable takes the value λ_i.

Before we go into the theory more, let's do an example. We will use the observable

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

for a qubit. This matrix has two eigenvectors,

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue 3,} \qquad \text{and} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ with eigenvalue 1.}$$

This means we can express M as a linear combination of projectors:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = 3 \,|\,+\rangle\langle+\,|\, + \,|\,-\rangle\langle-\,|\,.$$

One of the useful properties of the observable formulation is that we can easily calculate the expectation value of an observable applied to a quantum state.

Theorem 1 *Given a state $|\,\psi\rangle$ and an observable H, the expectation value of the measurement applied to H is*

$$\text{Expectation} = \langle\psi\,|\,H\,|\,\psi\rangle\,.$$

Before we prove this theorem, let's do an example. Suppose we have the state $|\,\psi\rangle = \frac{3}{5}\,|\,0\rangle + \frac{4}{5}\,|\,1\rangle$. The observable $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ measures it in the basis $\{|\,+\rangle\,, |\,-\rangle\}$, and returns the value 3 for the outcome $|\,+\rangle$ and the value 1 for the outcome $|\,-\rangle$.

The theorem gives

$$\langle\psi\,|\,M\,|\,\psi\rangle = \left(\frac{3}{5}, \frac{4}{5}\right) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix} = \frac{74}{25}$$

for the expectation vaue of the observable.

Now, the probability of the outcome $|\,+\rangle$ is $|\langle+|\psi\rangle|^2 = \frac{49}{50}$ and the probability of the outcome $|\,-\rangle$ is $|\langle-|\psi\rangle|^2 = \frac{1}{50}$. Thus, the expected value of the observable is

$$\frac{49}{50} \cdot 3 + \frac{1}{50} \cdot 1 = \frac{74}{25}.$$

We now prove the theorem. Suppose the eigenvalues of an observable H are λ_1, λ_2, $\ldots, \lambda_j$, and the corresponding eigenspaces are $S_1, S_2, \ldots, S_j$. Suppose further that Π_i is the orthogonal projector onto S_i. Let us now apply this observable to a quantum state $|\,\psi\rangle$. The outcome of the measurement is i with probability $\langle\psi\,|\,\Pi_i\,|\,\psi\rangle$. Thus, the expectation of the observable is

$$\begin{aligned}\sum_i \langle\psi\,|\,\Pi_i\,|\,\psi\rangle\,\lambda_i &= \langle\psi\,|\left(\sum_i \lambda_i \Pi_i\right)|\,\psi\rangle \\ &= \langle\psi\,|\,H\,|\,\psi\rangle\,,\end{aligned}$$

where the first equality holds because matrix multiplication is linear, and the second because $H = \sum_i \lambda_i \Pi_i$. This proves the theorem.