18.435/2.111 Homework # 7 Solutions

If you have the POVM with three elements:

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{1}{4} \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{3} & -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} & \frac{1}{4} \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

you can notice that the first two elements have rank 1 (they can be expressed as $|\,v\rangle\,\langle v\,|$ for some $|\,v\rangle$, and the third has rank 2. The first thing to do, in order to express this measurement as a projective measurement in a higher dimensional space, is to decompose the third element into a sum of two rank 1 matrices. There are many ways to do this, but possibly the most obvious one is $\frac{1}{3}\,|\,0\rangle\langle 0\,| + \frac{1}{2}\,|\,1\rangle\langle 1\,|$. We then have a four-element POVM which is a refinement of the original POVM, with elements $|\,v_i\rangle\langle v_i\,|$ where

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{3}}\,|\,0\rangle + \frac{1}{2}\,|\,1\rangle \\ v_2 &= \frac{1}{\sqrt{3}}\,|\,0\rangle - \frac{1}{2}\,|\,1\rangle \\ v_3 &= \frac{1}{\sqrt{3}}\,|\,0\rangle \\ v_4 &= \qquad\qquad \frac{1}{\sqrt{2}}\,|\,1\rangle \end{aligned}$$

We now put these vectors into a 2×4 matrix. The condition that $\sum_i |\,v_i\rangle\langle v_i\,|$ means that the two columns will be orthonormal.

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{2} \\ \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

We now have to extend this to a 4×4 unitary matrix, which we know we can do by Gram Schmidt orthogonalization. One way to do this is

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2} & \frac{1}{\sqrt{6}} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & -\frac{1}{2} & \frac{1}{\sqrt{6}} & -\frac{1}{2} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

The four row vectors then form a projective measurement which realizes the refinement of the desired POVM when restricted to the first two coordinates. To get the projections which realize the desired POVM, we must unrefine this projective measurement by taking $|\,v_3\rangle\langle v_3\,| + |\,v_4\rangle\langle v_4\,|$. The projections that realize the desired measurement are thus $|\,v_1\rangle\langle v_1\,|$, $|\,v_2\rangle\langle v_2\,|$, and $|\,v_3\rangle\langle v_3\,| + |\,v_4\rangle\langle v_4\,|$, where v_i is the ith row of the unitary matrix. Hence, the elements are:

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{1}{4} & \frac{1}{\sqrt{24}} & \frac{1}{4} \\ \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{24}} & \frac{1}{6} & \frac{1}{\sqrt{24}} \\ \frac{1}{\sqrt{12}} & \frac{1}{4} & \frac{1}{\sqrt{24}} & \frac{1}{4} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{3} & -\frac{1}{\sqrt{12}} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} & \frac{1}{4} & -\frac{1}{\sqrt{24}} & \frac{1}{4} \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{24}} & \frac{1}{6} & -\frac{1}{\sqrt{24}} \\ -\frac{1}{\sqrt{12}} & \frac{1}{4} & -\frac{1}{\sqrt{24}} & \frac{1}{4} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{3} & 0 & -\frac{\sqrt{2}}{3} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{\sqrt{2}}{3} & 0 & \frac{2}{3} & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

2: (Easy way) We have from Nielsen and Chuang that

$$|\,\psi_i\rangle\langle\psi_i\,| = \frac{1}{2}(I + \mathbf{v}_i \cdot \sigma)$$

We also have that

$$\sum q_i\,|\,\psi_i\rangle\langle\psi_i\,| = I$$

so taking trace of both sides,

$$\sum q_i = 2.$$

Now, we have

$$\begin{aligned} \sum q_i\,|\,\psi_i\rangle\langle\psi_i\,| &= \frac{1}{2}\sum q_i(I + \mathbf{v}_i \cdot \sigma) \\ &= I + \left(\sum_i q_i \mathbf{v}_i\right) \cdot \sigma \end{aligned}$$

but this must be I, so $\sum_i q_i \mathbf{v}_i = 0$.

You can also do it, without using this formula, by just looking at the x component of $\mathbf{v}$ (and then saying, by symmetry, that the y and z components also behave properly). This isn't hard, either, but it's not quite as straightforward.

3 This is pretty straightforward. If you measure the i'th element of the first POVM and the j'th element of the second, then $|\,\psi\rangle \to \sqrt{F_j}\sqrt{E_i}\,|\,\psi\rangle$, and the probability this happens is the product of the probability of obtaining the ith element of the first POVM times the probability of obtaining the jth element of the second, given you have the state resulting from the i element of the first POVM. This is

$$\langle\psi\,|\,E_i\,|\,\psi\rangle \cdot \frac{\langle\psi\,|\,\sqrt{E_i}F_j\sqrt{E_i}\,|\,\psi\rangle}{\langle\psi\,|\,E_i\,|\,\psi\rangle}$$

which is $\langle\psi\,|\,\sqrt{E_i}F_j\sqrt{E_i}\,|\,\psi\rangle$, as desired. To show this is a POVM, we take

$$M_{i,j} = \sqrt{F_i}\sqrt{E_j}$$

and calculate

$$\begin{aligned}\sum_{i,j} M_{i,j}^\dagger M_{i,j} &= \sum_{i,j} \sqrt{E_i}\sqrt{F_j}\sqrt{F_j}\sqrt{E_i} \\ &= \sum_i \sqrt{E_i}\left(\sum_j F_j\right)\sqrt{E_i} \\ &= \sum_i \sqrt{E_i} I \sqrt{E_i} = I,\end{aligned}$$

showing that the $M_{i,j}$ form a POVM.

4 This problem asks that if you have m linearly independent vectors, $|\,\psi_1\rangle, \ldots, |\,\psi_m\rangle$ in m dimensions, there is a POVM with $m+1$ outcomes such that if outcome E_i occurs, than Bob knows with certainty he has the state $|\,\psi_i\rangle$, and we also want to know that if the state is $|\,\psi_i\rangle$, there is some probability of outcome E_i.

We want that $\langle\psi_i\,|\,E_j\,|\,\psi_i\rangle = 0$ for all $i \neq j$, and $\langle\psi_i\,|\,E_i\,|\,\psi_i\rangle > 0$. Now, if you let $E_j = \alpha\,|\,v_j\rangle\langle v_j\,|$, $j = 1\ldots m$, what you need is that $\langle v_j|\psi_i\rangle = 0$ for $i \neq j$. However, all this is saying is that for $m-1$ vectors in m dimensions (all vectors but the jth one), there is a vector $|\,v_j\rangle$ which is perpendicular to all of them. We now need that this vector $|\,v_j\rangle$ is not perpendicular to $|\,\psi_j\rangle$. But it can't be, because if it were, we would have $|\,\psi_1\rangle \ldots |\,\psi_m\rangle$ all perpendicular to $|\,v_j\rangle$, so they would lie in an $m-1$ dimensional space (and thus couldn't be linearly independent).

Now, if we choose $E_j = \alpha\,|\,j\rangle\langle j\,|$, $j = 1..m$, we need for the completeness condition for the POVM ($\sum_j E_j = I$) that

$$E_{m+1} = I - \alpha \sum_{j=1}^{m} |\,j\rangle\langle j\,|$$

and that E_{m+1} is positive. I want to claim that this is satisfied if we take $\alpha = 1/m$. For if we do, for any $|\,\phi\rangle$

$$\begin{aligned}\langle\phi\,|\,E_{m+1}\,|\,\phi\rangle &= \langle\phi\,|\left(I - \alpha\sum_{j=1}^{m}|\,v_j\rangle\langle v_j\,|\right)|\,\phi\rangle \\ &= \langle\phi|\phi\rangle - \frac{1}{m}\sum_{j=1}^{m}\langle\phi|v_j\rangle\langle v_j|\phi\rangle \\ &\geq 1 - 1/m\sum_{j=1}^{m} 1 = 0.\end{aligned}$$

and thus E_{m+1} is positive.

5a Suppose we take $|\,v\rangle = \alpha\,|\,0\rangle + \beta\,|\,1\rangle$ and $|\,w\rangle = \alpha\,|\,0\rangle - \beta\,|\,1\rangle$, with α, β, real and $\alpha \geq \beta$. Since the vectors we pick are irrelevant and all that matters is the absolute value of their inner product, we can do this. The inner product is $r = \alpha^2 - \beta^2$. We want to choose POVM elements with elements E_1 proportional to $|\,\bar{w}\rangle\langle\bar{w}\,|$ and E_2 proportional to $|\,\bar{v}\rangle\langle\bar{v}\,|$, where $|\,\bar{w}\rangle$ and $|\,\bar{v}\rangle$ are the state orthogonal to $|\,w\rangle$ and $|\,v\rangle$, respectively.

We would now like to say that the constants of proportionality on E_1 and E_2 are the same. There's a simple symmetry argument that shows there is an optimal measurement which has them the same. Suppose that the optimal measurement has

$$E_1 = a\,|\,\bar{w}\rangle\langle\bar{w}\,|\,; \quad E_2 = b\,|\,\bar{v}\rangle\langle\bar{v}\,|\,; \quad E_3$$

By symmetry, there is another optimal measurement with

$$E_1 = b\,|\,\bar{w}\rangle\langle\bar{w}\,|\,; \quad E_2 = a\,|\,\bar{v}\rangle\langle\bar{v}\,|\,; \quad E_3'$$

But then consider the POVM

$$E_1 = \frac{a+b}{2}\,|\,\bar{w}\rangle\langle\bar{w}\,|\,; \quad E_2 = \frac{a+b}{2}\,|\,\bar{v}\rangle\langle\bar{v}\,|\,; \quad \frac{1}{2}(E_3 + E_3')$$

It is fairly easy to prove that this is also optimal, and we have the same constant of proportionality on E_1 and E_2.

Now, we will find the optimal constant of propotionality:

$$|\,\bar{v}\rangle\langle\bar{v}\,| + |\,\bar{w}\rangle\langle\bar{w}\,| = 2\beta^2\,|\,0\rangle\langle 0\,| + 2\alpha^2\,|\,1\rangle\langle 1\,|\,,$$

and when we normalize to make $|\,\bar{v}\rangle\langle\bar{v}\,| + |\,\bar{w}\rangle\langle\bar{w}\,| \le I$, we get that

$$E_1 + E_2 = \frac{\beta^2}{\alpha^2}\,|\,0\rangle\langle 0\,| + |\,1\rangle\langle 1\,|$$

So we get

$$E_3 = \frac{\alpha^2 - \beta^2}{\alpha^2}\,|\,0\rangle\langle 0\,|\,.$$

Note that E_3 is rank 1.

Now, the probability of outcome 3 is

$$\langle v\,|\,E_3\,|\,v\rangle = \alpha^2 - \beta^2 = r.$$

In a way, 2b and 2c were trick questions: both answers come out to r^2. What I meant to ask (it got misinterpreted by some) was: if you have two copies of the system which are in the same state (so either $|\,v\rangle \otimes |\,v\rangle$ or $|\,w\rangle \otimes |\,w\rangle$), which is better: making the same POVM twice or figuring out the optimal POVM to make on the combined system, which can be viewed as two vectors with an inner product of r^2. The answer is that it doesnt matter, the probability of not distinguising them is r^2 either way.