18.435/2.111 Homework # 3 Solutions

1: The density matrix is

$$|\,\psi\rangle\langle\psi\,| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{4} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}.$$

Taking Tr $_A$ gives

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix}.$$

Taking Tr $_B$ gives

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

Looking at Tr $_A$, one can see that the eigenvectors of $(1, \pm 1)$, which shows that the eigenvalues are $\frac{1}{2} \pm \frac{\sqrt{2}}{4}$.

An alternative way of calculating is

$$|\,\psi\rangle\langle\psi\,| = \frac{1}{4}(\sqrt{2}\,|\,00\rangle + |\,01\rangle + |\,11\rangle)(\sqrt{2}\,\langle 00\,| + \langle 01\,| + \langle 11\,|)$$

Taking Tr $_A$ gives

$$\frac{1}{4}(\sqrt{2}\,|\,0\rangle_B + |\,1\rangle_B)(\sqrt{2}\,{}_B\langle 0\,| + {}_B\langle 1\,|) \cdot {}_A\langle 0|0\rangle_A + \frac{1}{4}(|\,1\rangle_B\langle 1\,|) \cdot {}_A\langle 1|1\rangle_A$$

(I've left out the terms which vanish because they have $\langle 0|1\rangle$ and $\langle 1|0\rangle$ in them.)

2: When we take the partial trace of $|\,\psi\rangle\langle\psi\,|$ we get

$$\sum_{i,j} a_i a_j^* \,|\,v_i\rangle\,\langle v_j\,| \cdot \langle w_j|w_i\rangle.$$

Now, we know that this expression must be equal to

$$\sum_i \mu_i\,|\,v_i\rangle\langle v_i\,|\,.$$

However, equating coefficients on $|\,v_i\rangle\,\langle v_j\,|$, we see that this means that $\mu_i = a_i a_i^*$ and $\langle w_j|w_i\rangle = 0$ if $i \neq j$, showing that the $|\,w_j\rangle$ are an orthonormal basis. (We know they are unit vectors because we constructed them that way: the normalization was absorbed into a_i.)

3: We use the formula $\text{CNOT}_{A,B}\ \text{CNOT}_{B,A}\ \text{CNOT}_{A,B} = \text{SWAP}$. If we use the fact that a Toffoli gate is a controlled CNOT and a Fredkin gate is a controlled SWAP, we find that

$$\text{Toffoli}_{1,2,3}\ \text{Toffoli}_{1,3,2}\ \text{Toffoli}_{1,2,3} = \text{Fredkin}_{1,2,3}$$

where the indexes tell how the qubits fit into the Fredkin gate. (Note that I have defined my Fredkin gate with the qubits in a different order from Nielsen and Chuang, so for Nielsen and Chuang, you would have $\text{Fredkin}_{3,1,2}$ in the above formula.)

You can replace the outer two Tofoli gates with CNOT's by just checking that they work properly if if qubit 1 is $|\,0\rangle$ — if qubit 1 is $|\,1\rangle$, then the behavior is the same as the Toffoli. However, if qubit 1 is $|\,0\rangle$, the middle Toffoli and the Fredkin gate behave as the identity on the last two qubits, and we need to check that $\text{CNOT}^2 = \text{I}$, which is correct.

NC 4.28 I need to draw a picture for this ... I'll put it up later.

NC 4.31. All of these equations are straightforward to obtain by matrix multiplication.

You could save yourself a little work by using

$$C\sigma_y C = i\, C\sigma_x C \cdot C\sigma_z C$$

to obtain 4.33 from 4.32 and 4.34. You could also save yourself a little work by using 4.32, 4.33, 4.34, and 4.38 to obtain 4.35, 4.36, 4.37 and 4.39, respectively, by applying the identities $\text{H}\sigma_x\text{H} = \sigma_z$ and $\text{HC}_{1,2}\text{H} = \text{HC}_{2,1}\text{H}$.

NC 4.34.

Let $|\,v_+\rangle$ and $|\,v_-\rangle$ be the ± 1 eigenvectors of U. We can solve this by looking at the application of the circuit to the input step by step.

We start in the state $|\,0\rangle\,|\,\psi_{\text{in}}\rangle$. When we apply the first Hadamard, we obtain

$$2^{-1/2}(|\,0\rangle + |\,1\rangle)\,|\,\psi_{\text{in}}\rangle\,.$$

When we apply the gate U, we get

$$2^{-1/2}\,|\,0\rangle\,|\,\psi_{\text{in}}\rangle + |\,1\rangle\,(\alpha\,|\,v_+\rangle - \beta\,|\,v_-\rangle)$$

where $\alpha\,|\,v_+\rangle + \beta\,|\,v_-\rangle = |\,\psi_{\text{in}}\rangle$ is the decomposition of $|\,\psi_{\text{in}}\rangle$ into the eigenvectors of U. Now, this can be rewrriten as

$$2^{-1/2}\alpha(|\,0\rangle + |\,1\rangle)\,|\,v_+\rangle + 2^{-1/2}\beta(|\,0\rangle - |\,1\rangle)\,|\,v_-\rangle$$

The next Hadamard turns this into

$$\alpha\,|\,0\rangle\,|\,v_+\rangle + \beta\,|\,1\rangle\,|\,v_-\rangle\,,$$

and measuring the first qubit leaves the second qubit in an eigenstate of U.