18.435/2.111 Homework # 2 Solutions

1: First, notice that $RT = \omega TR$.

The state $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ works. We want to say that $R^i T^j \otimes I\,|\psi\rangle$ is orthogonal to $R^{i'} T^{j'} \otimes I\,|\psi\rangle$ if $i \neq i'$ or $j \neq j'$. Since RT commute at the cost of a phase, we can do this if we show that $\langle\psi\,|\,R^{i-i'}T^{j-j'}\,|\,\psi\rangle = 0$. Let's take the case of $j \neq j'$ first. Then notice $T \otimes I\,|\psi\rangle = \frac{1}{\sqrt{3}}(|10\rangle + |21\rangle + |02\rangle)$, which is perpendicular to $|\psi\rangle$ no matter what phases you put on the terms, and that all R does is apply phases to the terms. A similar argument works for for T^2.

Now, if we have $j = j'$, we need to show that $\langle\psi\,|\,R\,|\,\psi\rangle$ and $\langle\psi\,|\,R^2\,|\,\psi\rangle$ are 0. This is just because $R\,|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + \omega\,|11\rangle + \omega^2\,|22\rangle)$, and taking the inner product gives $\frac{1}{3}(1 + \omega + \omega^2) = 0$. (and similarly for R^2).

2: Suppose we let

$$
\begin{aligned}
|\psi\rangle &= a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle \\
\sigma_x \otimes I\,|\psi\rangle &= c\,|00\rangle + d\,|01\rangle + a\,|10\rangle + b\,|11\rangle \\
\sigma_z \otimes I\,|\psi\rangle &= a\,|00\rangle + b\,|01\rangle - c\,|10\rangle - d\,|11\rangle \\
i\sigma_y \otimes I\,|\psi\rangle &= c\,|00\rangle + d\,|01\rangle - a\,|10\rangle - b\,|11\rangle
\end{aligned}
$$

and we can calculate from the fact that these are orthonormal that we must have

$$
\begin{aligned}
aa^* + bb^* &= \frac{1}{2} \\
cc^* + dd^* &= \frac{1}{2} \\
ac^* + bd^* &= 0 \\
ca^* + db^* &= 0
\end{aligned}
$$

But this is exactly the condition that the rows of

$$\sqrt{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are orthonormal. However, if the rows of a matrix are orthonormal, the matrix is unitary and the columns are also orthonormal. The columns being orthonormal is easily checked to be the condition that

$$|\psi\rangle\,, \quad I \otimes \sigma_x\,|\psi\rangle\,, \quad I \otimes \sigma_y\,|\psi\rangle\,, \quad I \otimes \sigma_z\,|\psi\rangle$$

are orthogonal.

4a: Let's consider $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. The rest of the Bell states can be obtained by applying σ_b to $|\psi\rangle$, where b is one of x,y or z. Suppose that Alice and Bob apply H to both their qubits of $|psi\rangle$. We know that $|psi\rangle$ is invariant if the same Basis transformation is applied to both sides, so we get $|psi\rangle$. Now, since $H\sigma_x H = \sigma_z$, we have

$$\begin{aligned}(H \otimes H)(\sigma_x \otimes I)|\psi\rangle &= (\sigma_z \otimes I)(H \otimes H)|\psi\rangle \\ (\sigma_z \otimes I)|\psi\rangle &\end{aligned}$$

so $H \otimes H$ interchanges the Bell states $\sigma_x \otimes I|\psi\rangle$ and $\sigma_z \otimes I|\psi\rangle$. A similar argument shows $H \otimes H$ applies a -1 phase to $\sigma_y \otimes I|\psi\rangle$.

4b: The only permutations Alice can perform are those performed by σ_x, σ_z and σ_y. To see that, first note that her transformation U must take $|0\rangle \to |0\rangle$ and $|1\rangle \to |1\rangle$ or take $|0\rangle \to |1\rangle$ and $|1\rangle \to |0\rangle$. Otherwise, she would obtain both a $|00\rangle$ and a $|10\rangle$ term when applying U to $|00\rangle + |11\rangle$, and these terms don't simultaneously appear in any Bell state. By possibly multiplying by σ_x, we obtain a unitary that is diagonal. Now, by considering what happens when this unitary is applied to a Bell state, we realize that it must either be αI or $\alpha\sigma_z$, where α is an arbitrary complex phase.

4c: When we square

$$Q \otimes S + R \otimes S + R \otimes T - Q \otimes T$$

we get three kinds of terms. The first are those like $QQ \otimes SS = I$. The second are those like $RR \otimes ST$ which are canceled by a term such as $-QQ \otimes ST$. The third are terms of the form $QR \otimes ST$. There are four of these terms, and these add to $[Q, R] \otimes [S, T]$.

Now, we need to show that

$$\langle\psi|(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2|\psi\rangle$$

is at most 8, and taking the square root of this equation gives Tsirelsen's inequality. (This is because $Q \otimes S$ ... is an observable, and thus is diagonalizable, so

$$\langle\psi|(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)|\psi\rangle$$

is at most its largest eigenvalue.)

Thus, we need to show

$$\langle\psi|[Q, R] \otimes [S, T]|\psi\rangle \leq 4.$$

To do this, we can use the fact that the eigenvalues of a tensor product are the product of the eigenvalues, and so we need to show that $[Q, R]$ has eigenvalues of absolute value at most 2. But since

$$[Q, R] = QR - RQ$$

all we need do is show that QR has eigenvalues at most 1 if Q and R have eigenvalues ± 1. But Q and R have eigenvalues ± 1, so they are both Hermitian and unitary. We then see that QR is unitary (it isn't necessarily Hermitian), so that its eigenvalues are indeed of the form $e^i\theta$, θ real, and this shows that $QR - RQ$ is Hermitian and has eigenvalues between -2 and 2.