18.435/2.111 Homework # 1 Solutions

1a: I don't know any way to do this except multiply the whole thing out.

First,

$$j_x\sigma_x + j_y\sigma_y + j_z\sigma_z = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}.$$

Now, applying this to the vector

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

gives the vector

$$\begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ e^{i\phi}(\cos\frac{\theta}{2}\sin\theta - \sin\frac{\theta}{2}\cos\theta) \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

where the equality is obtained by applying trigonometric formula for sum. Thus, we have shown that this state is an eigenvector with eigenvalue 1.

To take care of the eigenvector with the eigenvalue -1, just notice that σ_j and $-\sigma_j$ have the same eigenvectors, but with opposite eigenvalues.

1b The direct way to do this is pretty straightforward. You can also use the fact that the eigenvalues of σ_j must be orthogonal, because σ_j is a Hermitian matrix, and by our calculation in 1a, the $+1$ eigenvalue of σ_j is the -1 eigenvalue of σ_{-j}.

1c If we multiply $\sigma_j\sigma_k$, we get, since $\sigma_x^2 = \sigma_y^2 = \sigma_z^2$,

$$\left(\sum_{l\in\{x,y,z\}} j_l k_l\right) I + \sum_{m\neq n\in\{x,y,z\}} j_m k_n \sigma_m \sigma_n$$

The first term is 0, since $\mathbf{j}\cdot\mathbf{k} = 0$, and looking at the second term, $\sigma_m\sigma_n = -\sigma_n\sigma_m$, implies that $\sigma_j\sigma_k = -\sigma_k\sigma_j$.

1d First we will prove it for states that are either perpendicular or equal to the j-axis. Now, if $\mathbf{j}$ and $\mathbf{k}$ are perpendicular, and we look at $\sigma_j\sigma_k\sigma_j$, this is $-\sigma_k$. This shows that σ_j interchanges the two eigenvectors of σ_k (because these eigenvectors have eigenvalues ± 1. Thus, if σ_k is perpendicular to σ_j, σ_j rotates an eigenvector of σ_k by 180° around the j-axis. It is easy to check that σ_j leaves its own eigenvectors invariant.

Now, if you know the absolute value of the inner product of two states on the Bloch sphere just depends on the angle between them, it is sufficient to show that σ_j does the right thing on its eigenvectors and on states perpendular to $\mathbf{j}$. How can we show this? The inner product of these two states squared is just $\frac{1}{4}\mathrm{Tr}\ (I+\sigma_j)(I+\sigma_k)$. (figure out why). So we need to show that Tr $\sigma_j\sigma_k$ depends only on the inner product between $\mathbf{j}$ and $\mathbf{k}$. But this is essentially the calculation we did in 1c.

2 We are trying to show that

$$\langle v \,|\, A \,|\, v\rangle + \langle \bar{v} \,|\, A \,|\, \bar{v}\rangle = \mathrm{Tr}\ A$$

One way to do it is to write A in the $|\, v\rangle\,, |\, \bar{v}\rangle$ basis. It is then obvious, and all the operations in the equation are invariant under unitary transformations. If you want to do it from first principles, you can write

$$|\, v\rangle = \alpha \,|\, 0\rangle + \beta \,|\, 1\rangle$$

$$|\, \bar{v}\rangle = \beta^* \,|\, 0\rangle - \alpha^* \,|\, 1\rangle$$

and the equation becomes

$$(\alpha^* \langle 0 \,| + \beta^* \langle 1 \,|)A(\alpha \,|\, 0\rangle + \beta \,|\, 1\rangle) + (\beta \langle 0 \,| - \alpha \langle 1 \,|)A(\beta^* \,|\, 0\rangle - \alpha^* \,|\, 1\rangle)$$

and if you expand all the terms and cancel, you get

$$\langle 0 \,|\, A \,|\, 0\rangle + \langle 1 \,|\, A \,|\, 1\rangle\,,$$

which is Tr A.

3a We have

$$J_z^2 = \tfrac{1}{2}(\sigma_z \otimes \sigma_z + I \otimes I)$$

The second term commutes with everything, so the only thing to check is that the first terms of J_x^2, J_y^2, and J_z^2 all commute. Since $\sigma_x\sigma_z = -\sigma_z\sigma_x$, we have

$$\begin{aligned}(\sigma_x^A \otimes \sigma_x^B)(\sigma_z^A \otimes \sigma_z^B) &= \sigma_x^A\sigma_z^A \otimes \sigma_x^B\sigma_z^B \\ &= (-\sigma_z^A\sigma_x^A) \otimes (-\sigma_z^B\sigma_x^B) \\ &= (-1)^2(\sigma_z^A \otimes \sigma_z^B)(\sigma_x^A \otimes \sigma_x^B),\end{aligned}$$

and similarly for the other pairs.

Here, I've put in the A and B subscripts telling which of the qubits the Pauli matrix belongs with to try to make it clearer, but you can figure out which is which by looking at their positions in the tensor product.

3b First ,note that $\sigma_x \otimes I$, $\sigma_y \otimes I$ and $\sigma_z \otimes I$ either commute or anticommute with J_x^2, J_y^2 and J_z^2. If operators F and G anticommute (or commute), then if $|\, \psi\rangle$ is an eigenvector of F, it will be the case that $G \,|\, \psi\rangle$ is also an eigenvector of F. This is because, if λ is the eigenvalue for $|\, \psi\rangle$, then

$$FG \,|\, \psi\rangle = -GF \,|\, \psi\rangle = -\lambda G \,|\, \psi\rangle\,.$$

The other eigenvectors are thus $(\sigma_b \otimes I) \,|\, \psi\rangle$ for $b = x, y, z$, i.e., the Bell states.

4a

$$
\begin{aligned}
J_x &= \tfrac{1}{2}(\sigma_x \otimes I + I \otimes \sigma_x) \\
&= \frac{1}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

We now need to change basis to merge the middle two rows using

$$
|\,0_z\rangle = \frac{1}{\sqrt{2}}(|\,10\rangle + |\,01\rangle)
$$

This gives

$$
J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

Similarly,

$$
J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}
$$

The eigenvectors for J_x are (unnormalized) $(1, 0, -1)$ for the eigenvalue 0, and $(1, \pm\sqrt{2}, 1)$ for the eigenvalues ± 1.

4b We calculated the eigenvectors of J_x in 4a. We now need to project the first qutrit of the entangled state onto these eigenvectors, and obtain the resulting state of the second qubit. For the +1 eigenvector (now normalized), we have

$$
\begin{aligned}
\frac{1}{\sqrt{3}}\langle\uparrow_x|\,(|\uparrow_z\downarrow_z\rangle - |\,0_z0_z\rangle + |\downarrow_z\uparrow_z\rangle) &= \frac{1}{2\sqrt{3}}(\langle\uparrow_z| + \sqrt{2}\,\langle 0_z| + \langle\downarrow_z|)(|\uparrow_z\downarrow_z\rangle - |\,0_z0_z\rangle + |\downarrow_z\uparrow_z\rangle) \\
&= \frac{1}{2\sqrt{3}}(|\downarrow_z\rangle - \sqrt{2}\,|\,0_z\rangle + |\uparrow_z\rangle) \\
&= \frac{1}{\sqrt{3}}\,|\downarrow_x\rangle,
\end{aligned}
$$

and since this state has length $\frac{1}{\sqrt{3}}$, the probabiity is $\frac{1}{3}$. For the -1 eigenvector, we similarly have

$$
\frac{1}{\sqrt{3}}\langle\downarrow_x|\,(|\uparrow_z\downarrow_z\rangle - |\,0_z0_z\rangle + |\downarrow_z\uparrow_z\rangle) = \frac{1}{\sqrt{3}}\,|\uparrow_x\rangle
$$

For the 0 eigenvector, we have

$$
\frac{1}{\sqrt{6}}(\langle\uparrow_z| - \langle\downarrow_z|)(|\uparrow_z\downarrow_z\rangle - |\,0_z0_z\rangle + |\downarrow_z\uparrow_z\rangle) = \frac{1}{\sqrt{6}}(|\downarrow_z\rangle - |\uparrow_z\rangle) = -\frac{1}{\sqrt{3}}\,|\,0_x\rangle
$$

4c It is easy to calculate that $J^2 = 2I$. Since J_x^2, J_y^2 and J_z^2 all commute, they are simultaneously observable, and their sum must be 2. Since each of these operators must be 1 or 0, two will have the value 1 and one 0.