18.435/2.111 Homework # 9

Due Tuesday, December 9.

1: Show that Alice can teleport two qubits "through" the gate

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

in the following manner.

Suppose Alice and Bob share four qubits in the state

$$\frac{1}{2}\left(|\,0000\rangle + |\,0101\rangle + |\,1010\rangle - |\,1111\rangle\right)$$

with Alice holding the first two qubits and Bob holding the last two. Alice has two qubits in unknown (to her) states $|\,\phi\rangle$ and $|\,\psi\rangle$. She makes a joint measurement on the qubit in state $|\,\phi\rangle$ and her first entangled qubit using the Bell basis, and she also measures her second qubit $|\,\psi\rangle$ and her second entangled qubit using the Bell basis. Alice sends the results of both measurements to Bob over a classical channel, and Bob applies, to his halves of the two EPR pairs, Pauli matrices which depend on the classical bits he received. Show that if he applies the correct Pauli matrices, Bob will end up with $S\,|\,\phi\rangle\,|\,\psi\rangle$. How do the Paulis Bob applies depend on the results of Alice's measurements?

2: Suppose we have two classical linear codes $C_2 \subset C_1$. Consider the state

$$\frac{1}{|C_2|^{1/2}} \sum_{c \in C_2} |\,v + c\rangle\,(-1)^{c \cdot t}$$

where $v \in C_1$ and t is an arbitrary vector. Suppose we apply the Hadamard transformation on every qubit. What state do we obtain? Show that it is in the CSS code translated by s, i.e., it is a superposition of codewords

$$\left|\,w + C_1^{\perp} + s\right\rangle = \frac{1}{|C_1^{\perp}|^{1/2}} \sum_{c \in C_1^{\perp}} |\,w + c + s\rangle$$

with $w \in C_2^{\perp}$. How does s depend on t?

3a: Consider the modification to Grover's algorithm where the oracle now performs

$$\begin{aligned} O\,|\,x\rangle &= e^{i\phi}\,|\,x\rangle \quad \text{if } x \text{ is a target state} \\ O\,|\,x\rangle &= |\,x\rangle \quad \text{otherwise.} \end{aligned}$$

Show that if you use the transformation

$$\tilde{G} = H^{\otimes n}\left[(1 - e^{i\phi})\,|\,0\rangle\langle 0\,| - I\right] H^{\otimes n} O$$

instead of the standard Grover iteration, for any state with M/N sufficiently large you can choose ϕ so the algorithm finds a target state with probability 1 after one iteration.

3b: For what values of M/N is there such a ϕ? How can you combine the results of 3a with the standard Grover iteration (using the standard Grover oracle) to obtain a modification of Grover's algorithm that finds a marked state with probability 1, given that M/N is known?

The next three problems deal with approximate cloning. We have seen that we cannot perfectly clone a quantum state. However, there are approximate cloners that work moderately well. The optimal cloner for taking one qubit to two qubits is the transformation

$$\begin{aligned} |\,0\rangle &\rightarrow \sqrt{\frac{2}{3}}\,|\,00\rangle_{AB}\,|\,0\rangle_R + \sqrt{\frac{1}{6}}(|\,01\rangle_{AB} + |\,10\rangle_{AB})\,|\,1\rangle_R \\ |\,1\rangle &\rightarrow \sqrt{\frac{2}{3}}\,|\,11\rangle_{AB}\,|\,1\rangle_R + \sqrt{\frac{1}{6}}(|\,01\rangle_{AB} + |\,10\rangle_{AB})\,|\,0\rangle_R\,, \end{aligned}$$

where A and B are clones of the original qubit, and R is a reference system.

4: Show that if we call the cloning transfomation C, then for any pure state $|\,\psi\rangle$ on one qubit,

$${}_A\langle\psi\,|\,\mathrm{Tr}_{BR}\left(C\,|\,\psi\rangle\,\langle\psi\,|\,C^\dagger\right)|\,\psi\rangle_A = \frac{5}{6}$$

so this transformation clones with *fidelity* 5/6.

5: If we trace out the reference system R, find Krauss operators E_1 and E_2 so that the cloning transformation is expressible in the operator sum formalism as

$$\rho \rightarrow \sum_i E_i \rho E_i^\dagger.$$

Note that the matrices E_i will not be square.

6a: Now, suppose we change bases by taking

$$F_i = U^{\otimes 2} E_i U^{\dagger}$$

where

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

What are the F_i?

6b: Show that the cloning operation is invariant under this change of basis by finding a unitary matrix $\{u_{ij}\}$ such that

$$F_i = \sum_j u_{ij} E_j.$$