18.435/2.111 Homework # 8

Due Thursday, November 25.

1: Do Exercise 10.25 in Nielsen and Chuang.

2: Do Exercise 10.27 in Nielsen and Chuang.

Suppose you encode the qubit $|\,\psi\rangle = \alpha\,|\,0\rangle + \beta\,|\,1\rangle$ using the 3-qubit code which can correct any one σ_x error. The next two problems will be about this code.

3: Suppose somebody measures one of the qubits in the $|\,0\rangle$, $|\,1\rangle$ basis. What happens when you try to correct the error?

4: Suppose somebody measures one of the qubits in the $\frac{1}{\sqrt{2}}(|\,0\rangle \pm |\,1\rangle)$ basis. What happens when you try to correct the error?

5: Now suppose the amplitude damping channel with damping parameter p is applied to one of the qubits, and you apply the correction procedure. The resulting state can be expressed as the encoding of a density matrix. What is this matrix?

6: Write the Krauss operator elements for the amplitude damping channel (i.e., the A_j in the expression $\rho \rightarrow \sum_j A_j \rho A_J^\dagger$) as a linear combination of 1, σ_x, σ_y, σ_z. What is the connection to the solution of the previous problem.

7: (You should not need tedious computation to do this if you approach it the right way.) Suppose you have the state $\alpha\,|\,0\rangle + \beta\,|\,1\rangle$ encoded with the three-qubit phase-error-correcting code, and the amplitude damping channel with damping parameter p is applied to one of the qubits. What is probability of error after applying the correction procedure?

8: Suppose you are trying to distinguish between states $|\,v_1\rangle \ldots |\,v_n\rangle$. The "pretty good measurement" is defined as follows. Let

$$\mu = \sum_{i=1}^{n} |\,v_i\rangle\langle v_i\,|\,.$$

The pretty good measurement has POVM elements

$$\mu^{-1/2}\,|\,v_i\rangle\langle v_i\,|\,\mu^{-1/2}$$

Prove that the pretty good measurement is actually a POVM.

9: It can be shown that if the $|\, v_i \rangle$'s are given with equal probability, the probability of error obtained using the pretty good measurement is no more than twice the best error probability possible using any POVM. Suppose you are trying to distinguish between set of $p^2 + p$ states which consist of the p^2 states

$$\frac{1}{\sqrt{p}} \sum_{x=0}^{p-1} |\, x \rangle \,|\, ax + b \rangle$$

as well as the p states

$$\frac{1}{\sqrt{p}} \sum_{x=0}^{p-1} |\, a \rangle \,|\, x \rangle \,,$$

for $a, b \in \{0, \ldots, p-1\}$. What is the error probability for the pretty good measurement? **Hint:** μ can be expressed as $\alpha I + \beta \,|\, w \rangle\langle w \,|$ for some state $|\, w \rangle$.

10: Suppose you are trying to measure quantities a, b for a state chosen from some ensemble of quantum states $|\, \psi(a, b) \rangle$. Now, suppose you have one quantum algorithm which, for any of these states, returns a with probability $1 - \epsilon_1$, and a different quantum algorithm which returns b with probability $1 - \epsilon_2$. If ϵ_1 and ϵ_2 are small, show that there is a quantum algorithm which measures both a and b simultaneously with success probability near 1.

Note: I have had several people ask me to clarify what (10) is asking. You should first note that the $|\, \psi(a, b) \rangle$ are not mutually orthogonal. (This would make the problem easier.)

What you are given is that there is a quantum circuit whose input takes $|\, \psi \rangle$ in the first register of k qubits, and $|\, 0 \rangle$ in the remaining qubits. At the output of the quantum circuit, if the input was $|\, \psi(a, b) \rangle$, and you measure the first k' qubits, the probability that you observe $|\, a \rangle$ in them is at least $1 - \epsilon_1$, and similarly for $|\, b \rangle$ with probability $1 - \epsilon_2$. Now, show that there is a quantum circuit which, for input $|\, \psi(a, b) \rangle$, will output both a and b with some probability which is close to 1 if ϵ_1 and ϵ_2 are small.