18.435/2.111 Homework # 7

Due Thursday, October 30.

1: Suppose you have a POVM with the three elements

$$E_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{1}{4} \end{pmatrix} \quad E_2 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{\sqrt{12}} \\ -\frac{1}{\sqrt{12}} & \frac{1}{4} \end{pmatrix} \quad E_3 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Show how to perform this by first embedding the 2-dimensional space into a 4-dimensional space and then performing a von Neumann measurement.

2: Suppose that if you have a 2-dimensional POVM with rank-1 outcomes

$$q_1 \,|\, \psi_1 \rangle\langle \psi_1 \,| \,, q_2 \,|\, \psi_2 \rangle\langle \psi_2 \,| \,, \ldots, q_k \,|\, \psi_k \rangle\langle \psi_k \,| \,,$$

where $|\, \psi_i \rangle$ are unit vectors. Consider the vectors $\mathbf{v}_i$ on the Bloch sphere corresponding to $|\, \psi_i \rangle$. Show that

$$\sum_i q_i \mathbf{v}_i = 0.$$

3: Suppose that you have two POVM's, specified by $\{E_i\}$ and $\{F_j\}$, with $\sum_i E_i = \sum_j F_j = I$. The first POVM maps

$$|\, \psi \rangle \rightarrow \sqrt{E_i}\,|\, \psi \rangle$$

with probability $\langle \psi \,|\, E_i \,|\, \psi \rangle$. The second POVM maps

$$|\, \psi \rangle \rightarrow \sqrt{F_j}\,|\, \psi \rangle$$

with probability $\langle \psi \,|\, F_j \,|\, \psi \rangle$. Show that when you first apply E_i and second apply F_j you obtain a POVM. (That is, $|\, \psi \rangle \rightarrow M_k \,|\, \psi \rangle$ with $\sum M_k^\dagger M_k = I$. What are its elements?

4: Problem 2.64 in NC.

5a: Suppose you have two quantum states $|\, v \rangle$ and $|\, w \rangle$, whose inner product is r. There are 3-outcome POVM's such that outcome one implies the state was $|\, v \rangle$, outcome two implies the state was $|\, w \rangle$, and outcome 3 leaves you uncertain what the state was. What is the minimum probability of outcome 3, expressed in terms of r?

NOTE: (Without loss of generality you may assume that r is real and you may choose specific $|\, v \rangle$ and $|\, w \rangle$ with $\langle v | v \rangle = r$. This may help in the calculations.)

5b: (Easy, given 5a) Suppose you have two copies of the state in 5a. One can apply the POVM you found in part 5a to each of the copies. Now what is the probability that you are uncertain about the state.

5c: Suppose you consider the states $|\, v \rangle \otimes |\, v \rangle$ and $|\, w \rangle \otimes |\, w \rangle$ to be states $|\, v' \rangle$ and $|\, w' \rangle$ with inner product r^2, and apply the POVM from part 5a for two states with inner product r^2. What is the probability of uncertainty about the state after you apply the POVM? How does this compare with the answer you got in 5b? When are the two quantities equal?