18.435/2.111 Homework # 3

Due Thursday, October 2.

1: Consider the state

$$|\,\psi\rangle_{AB} = \tfrac{1}{2}(\sqrt{2}\,|\,00\rangle + |\,01\rangle + |\,11\rangle).$$

Find $\rho_B = \text{Tr}_A\,|\,\psi\rangle\,\langle\psi\,|$ and $\rho_A = \text{Tr}_B\,|\,\psi\rangle\,\langle\psi\,|$. Show that these matrices have the same eigenvalues by showing $\text{Tr}\,\rho_A^2 = \text{Tr}\,\rho_B^2$. What are these eigenvalues?

2 we will now prove the property we found in problem 1 holds in general.

Suppose we have a pure state $|\,\psi\rangle$ on a tensor product state space $A \otimes B$. We know $\rho = \text{Tr}_B\,|\,\psi\rangle\,\langle\psi\,|$ is a Hermitian matrix, and thus diagonalizable, so $\rho = \sum_i \mu_i\,|\,v_i\rangle\,\langle v_i\,|$ for some orthonormal basis $|\,v_i\rangle$. Now, since $|\,v_i\rangle$ is a basis, we have that ψ can be expressed as

$$|\,\psi\rangle = \sum a_i\,|\,v_i\rangle \otimes |\,w_i\rangle$$

for some unit vectors $|\,w_i\rangle$ in B. Take the partial trace of the above expression for $|\,\psi\rangle\,\langle\psi\,|$ and use the results to show that the $|\,w_i\rangle$ are orthonormal, that is, $\langle w_j|w_i\rangle = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker δ function.

By incorporating the phase of a_i in $|\,w_i\rangle$, we get the Schmidt depomposition for a pure state: every pure state can be written as

$$|\,\psi\rangle = \sum \sqrt{\mu_i}\,|\,v_i\rangle \otimes |\,w_i\rangle\,.$$

3 The Fredkin gate, which operates on three bits (or qubits) is a controlled SWAP. That is, you interchange bits two and three if bit one is 1, and do nothing if bit one is 0. Show how to build a Fredkin gate out of three Toffoli gates. Show how to build a Fredkin gate out of one Toffoli gate and several CNOT gates.

In Nielsen and Chuang, do problems 4.28, 4.31, 4.34, and 4.35. For 4.28, I believe you will have to use CNOT and/or Toffoli gates as well as the C-V and C-$V^\dagger$ gates mentioned in the problem.