18.435/2.111 Homework # 2

Due Thursday, September 25.

1: Consider the matrices

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

where ω is a cube root of 1. Find a state $|\psi\rangle$ on the tensor product of two qutrits so that the nine quantum states $(R^i T^j \otimes I)|\psi\rangle$, $0 \le i, j \le 2$ are all orthogonal. Explain how this lets Alice and Bob perform superdense coding, and send two classical trits at the price of the transmission of one quantum trit, and the use of one copy of $|\psi\rangle$?

2: Suppose you have a quantum state $|\psi\rangle$ on a tensor product of two qubits. Suppose that the four states $|\psi\rangle$, $\sigma_x \otimes I|\psi\rangle$, $\sigma_y \otimes I|\psi\rangle$, $\sigma_z \otimes I|\psi\rangle$ are all orthogonal. Show that the four states $|\psi\rangle$, $I \otimes \sigma_x|\psi\rangle$, $I \otimes \sigma_y|\psi\rangle$, $I \otimes \sigma_z|\psi\rangle$ are all orthogonal.

3: Give a sequence of one-qubit gates and CNOT gates which, applied to two qubits, transforms

$$|00\rangle \to \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
$$|11\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

What does your circuit do to the two states $|01\rangle$ and $|10\rangle$? What could an arbitrary such circuit do to these states?

4a Recall the four Bell states were

$$\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \qquad \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

If Alice and Bob apply a Hadamard gate H to both sides of a Bell state, show that two of the Bell states are interchanged, and two of the states are left unchanged by this transformation.

4b Suppose Alice tries to permute the four Bell states by applying a unitary transformation to her qubit (and Bob applies the identity). What permutations can she perform (for a permutation, note that every Bell state must be mapped to a Bell state)? Give a proof of your answer.

5 Tsirelsen's inequality. This is Nielsen and Chuang, problem 2.3. Suppose Q, R, S and T are observables on a single qubit, each having two eigenvalues of $\{-1, 1\}$. Prove that

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T],$$

where

$$[A, B] = AB - BA$$

is the *commutator* of A and B. Use this result to prove that the expectation

$$\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle \leq 2\sqrt{2}$$

What I want is for you to show that for any $|\,\psi\rangle$, the expectation $\langle \psi \,|\, M \,|\, \psi \rangle$ is at most $2\sqrt{2}$, where M is the observable give above. This shows that the amount of violation of Bell's inequality in the example given in class (and in Nielsen and Chuang) is tight.