18.435/2.111 Homework # 1

Due Thursday, September 18.

1a: The quantum state

$$\cos\frac{\theta}{2}\,|\,0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|\,1\rangle$$

corresponds to the point on the Bloch sphere

$$\mathbf{j} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Show that the operator

$$\sigma_{\mathbf{j}} = j_x\sigma_x + j_y\sigma_y + j_z\sigma_z$$

has eigenvectors which correspond to the points $\pm\mathbf{j}$ on the Bloch sphere.

1b: Show that antipodal points on the Bloch sphere are orthogonal quantum states.

1c: Show that if the vectors $\mathbf{j}$ and $\mathbf{k}$ are perpendicular, then $\sigma_{\mathbf{j}}$ and $\sigma_{\mathbf{k}}$ anticommute.

1d: Show that applying $\sigma_{\mathbf{j}}$ to a quantum state $|\,\psi\rangle$ rotates $|\,\psi\rangle$ by the angle π around the $\mathbf{j}$-axis on the Bloch sphere.
Hint: one way to do it is use 1a, 1b, 1c. (There are more straightforward, although calculation-intensive, ways to do it, so this hint may not be of much use).

2: Suppose you have two orthogonal states of a qubit, $|\,v\rangle$ and $|\,\bar{v}\rangle$, and an observable A on the qubit. Show that if you add the expectation value of the observable A for the state $|\,v\rangle$ to the expectation value for the state $|\,\bar{v}\rangle$, you obtain $\mathrm{Tr}A$.

3: Recall that I said (assuming $\hbar = 1$) that $\frac{1}{2}\sigma_z$ was an observable for angular momentum in the z direction (similarly for x and y). If you have two qubits, then the observable for total angular momentum in the z direction is

$$J_z = \frac{1}{2}(\sigma_z \otimes I + I \otimes \sigma_z)$$

and similarly for x and y.

3a: Show that J_x^2, J_y^2 and J_z^2 all commute.

3b: Since they are commuting, they have simultaneous eigenvectors. You already know one of these eigenvectors: it is the state with 0 angular momentum $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$. What are the other three simultaneous eigenvectors?

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4: You can work out the behavior of a spin-1 particle by considering the behavior of two spin-1/2 particles that are guaranteed not to be in the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$ (this state has $J_x = J_y = J_z = 0$). If we work in the basis

$$\{|\uparrow_z\rangle, |\,0_z\rangle, |\downarrow_z\rangle\} = \left\{|\,00\rangle, \tfrac{1}{\sqrt{2}}(|\,01\rangle + |\,10\rangle), |\,11\rangle\right\}$$

then

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

4a: Calculate the operators J_x and J_y in this basis and find their eigenvectors.

4b: Suppose you have an entangled pair of spin-1 particles in the state

$$\frac{1}{\sqrt{3}}(|\uparrow_z\downarrow_z\rangle - |\,0_z 0_z\rangle + |\downarrow_z\uparrow_z\rangle)$$

and you measure the first one in the $\{\,|\uparrow_x\rangle, |\,0_x\rangle, |\downarrow_x\rangle\,\}$ basis. What are the probabilities for each outcome, and what is the state of the second particle after the measurement? Here, the states $|\uparrow_x\rangle, |\,0_x\rangle, |\downarrow_x\rangle$ stand for the eigenvectors with $+1$, 0, -1 spin along the x axis, and similarly for z.

4c: What is $J^2 = J_x^2 + J_y^2 + J_z^2$. Use the answer to explain why, if these three operators are measured for a spin-1 particle, then J_α^2 will be 0 in one of the three directions, and 1 in the other two directions.

This is related to the Kochen-Specker theorem. There is a set of 33 directions (the number started out much larger, and has also been reduced still further) in three dimensions, which contains a number of triples of orthogonal directions (i.e., bases). This set of directions cannot be colored red and green so that in every such triple, exactly one is colored green, and such that no pair of orthogonal directions are both colored green. This proves that there is no way of assigning "hidden variables" to the quantum measurements of $J_{\mathbf{k}}^2$ along these 33 directions $\mathbf{k}$ so that the predictions of quantum mechanics are deterministically satisfied.

For those of you who are interested, these directions are (unnormalized)

- $(1, 0, 0)$ and symmetries [3 vectors in total]
- $(1, \pm 1, 0)$ and symmetries [6 vectors in total]
- $(\sqrt{2}, \pm 1, 0)$ and symmetries [12 vectors in total]
- $(\sqrt{2}, \pm 1, \pm 1)$ and symmetries [12 vectors in total].