18.435/2.111 Homework 9 Solutions

1: First, three facts that follow from the definitions:

$$
\begin{aligned}
e^{i\frac{\omega t}{2}\sigma_z} &= \cos(\frac{\omega t}{2})I + i\sin(\frac{\omega t}{2})\sigma_z &(1)\\
\sigma_\pm\sigma_z &= (\sigma_x \pm i\sigma_y)\sigma_z = -i\sigma_y \mp \sigma_x = \mp\sigma_\pm &(2)\\
\sigma_z\sigma_\pm &= \pm\sigma_\pm &(3)
\end{aligned}
$$

We can use these to show the desired result:

$$
\begin{aligned}
e^{i\frac{\omega t}{2}\sigma_z}\sigma_\pm e^{-i\frac{\omega t}{2}\sigma_z} &= (\cos(\frac{\omega t}{2})I + i\sin(\frac{\omega t}{2})\sigma_z)\sigma_\pm(\cos(\frac{\omega t}{2})I - i\sin(\frac{\omega t}{2})\sigma_z)\\
&= (\cos(\frac{\omega t}{2}) \pm i\sin(\frac{\omega t}{2}))(\cos(\frac{\omega t}{2}) \pm i\sin(\frac{\omega t}{2}))\sigma_\pm &(4)\\
&= (e^{\pm i\frac{\omega t}{2}})^2\sigma_\pm &(5)\\
&= e^{\pm i\omega t}\sigma_\pm. &(6)
\end{aligned}
$$

2: Let's write the two gates as U_1 and U_2:

$$
\begin{aligned}
U_1 &= |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R_y(\theta) &(7)\\
U_2 &= I \otimes |0\rangle\langle 0| + \sigma_x \otimes |1\rangle\langle 1|, &(8)
\end{aligned}
$$

where $R_y(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}.\end{bmatrix}$ Let's also remember that we can write the density matrix in terms of its matrix elements:

$$\rho = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|, \qquad (9)$$

where $\rho_{ij} = \langle i|\rho|j\rangle$. After the first gate, we have

$$
\begin{aligned}
U_1(\rho\otimes|0\rangle\langle 0|)U_1^\dagger &= \rho_{00}|0\rangle\langle 0|\otimes|0\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|\otimes R_y(\theta)|0\rangle\langle 0|R_y(\theta)^\dagger\\
&= \rho_{00}|00\rangle\langle 00|\\
&+ \rho_{11}|1\rangle\langle 1|\otimes(\cos\frac{\theta}{2})|0\rangle + \sin\frac{\theta}{2}|1\rangle)(\cos\frac{\theta}{2})\langle 0| + \sin\frac{\theta}{2}\langle 1|).
\end{aligned}
$$

The cross-terms go away when we look at the result after the second gate:

$$U_2U_1(\rho\otimes|0\rangle\langle 0|)U_1^\dagger U_2^\dagger = \rho_{00}|00\rangle\langle 00| + \cos^2\frac{\theta}{2}\rho_{11}|10\rangle\langle 10| + \sin^2\frac{\theta}{2}\rho_{11}|01\rangle\langle 01|$$

After we measure the second qubit (equivalent to taking the partial trace of the second qubit), we have

$$\rho_{00}|0\rangle\langle 0| + \cos^2\frac{\theta}{2}\rho_{11}|1\rangle\langle 1| + \sin^2\frac{\theta}{2}\rho_{11}|0\rangle\langle 0| = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger, \qquad (10)$$

where $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \cos\frac{\theta}{2} \end{bmatrix}$ and $E_1 = \begin{bmatrix} 0 & \sin\frac{\theta}{2} \\ 0 & 0 \end{bmatrix}$. Note that this is the amplitude damping channel with $\gamma = \sin^2\frac{\theta}{2}$.

3: Several of you noticed that this problem, as stated in Nielsen and Chuang, is incorrect. The reasoning is sound, but there is an error in the specification of $\{E_i^{dr}\}$. I will explain in the course of the problem.

Our initial density matrix is given by

$$|\psi\rangle\langle\psi| = |a|^2 |01\rangle\langle 01| + ab^* |01\rangle\langle 10| + a^*b |10\rangle\langle 01| + |b|^2 |01\rangle\langle 01| . \quad (11)$$

When we apply the amplitude damping channel, most of the terms cancel (As $E_1|0\rangle = 0$) and we get:

$$\begin{aligned} \mathcal{E}_{AD} \otimes \mathcal{E}_{AD}(|\psi\rangle\langle\psi|) &= (1-\gamma)(|\psi\rangle\langle\psi|) + \gamma(|a|^2 + |b|^2)|00\rangle\langle 00| \\ &= (1-\gamma)\rho + \gamma|00\rangle\langle 00| . \end{aligned} \quad (12)$$

Let's interpret this result. With probability $1-\gamma$, the state is passed through unperturbed. With probability γ, the state is replaced by $|00\rangle\langle 00|$.

Here is where the problem is stated incorrectly. In order to describe this operation, we actually need three operator elements $\{E_0^{dr} = \sqrt{1-\gamma}I, E_1^{dr} = \sqrt{\gamma}|00\rangle\langle 01|, E_3^{dr} = \sqrt{\gamma}|00\rangle\langle 10|\}$. Notice, also, that we are only defining the action on the subspace of interest. Extra credit to those of you who spotted this error!

4: To simplify the notation, let's work in a basis where $|v\rangle = |0\rangle$ and $|w\rangle = a|0\rangle + b|1\rangle$. Notice that the non-orthogonality constraints imply $a \neq 0$ and $b \neq 0$.

Let's write Φ with the operator elements $\{E_i\}$. As we discussed in the solutions to homework 8, we can interpret a pure state $|\psi\rangle$ input to this operation as resulting in states $E_i|\psi\rangle / \|E_i|\psi\rangle\|$ with probability $\|E_i|\psi\rangle\|^2$. Thus, if the output state is also pure, we know that $E_i|\psi\rangle \propto E_j|\psi\rangle$ for all i, j.

What does this tell us about $|v\rangle$ and $|w\rangle$? Since each are mapped to themselves, we can conclude that $E_i|0\rangle = c_i e^{i\theta}|0\rangle$ and $E_i|w\rangle = d_i e^{i\phi}|w\rangle$. But note that $E_i|w\rangle = aE_i|0\rangle + bE_i|1\rangle \Rightarrow ac_i e^{i\theta}|0\rangle + bE_i|1\rangle = d_i e^{i\phi}(a|0\rangle + b|1\rangle$. Clearly, this can only happen if $d_i = c_i$ and $\theta = \phi$. This further implies that $E_i|1\rangle = c_i e^{i\theta}|1\rangle$.

We have now established the action of $\Phi(\cdot)$ on the pure state basis $|0\rangle$ and $|1\rangle$. This tells us that $\Phi(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$ for all $|\psi\rangle$. Since any ρ can be written in the form $\rho = \sum_k p_k |k\rangle\langle k|$, by linearity we see that Φ is the identity:

$$\Phi(\rho) = \Phi(\sum_k p_k |k\rangle \langle k|) \tag{13}$$

$$= \sum_k p_k \Phi(|k\rangle \langle k|) \tag{14}$$

$$= \sum_k p_k |k\rangle \langle k| \tag{15}$$

$$= \rho. \tag{16}$$

5a: We want a measurement with two outcomes given by Π_1 and Π_2. Let's define outcome 1 as indicating our hypothesis that ρ_1 was given, and outcome 2 indicating ρ_2. How do we write down the probability of error? This is stated quite succinctly by conditional probabilities (though I'm abusing notation here a bit):

$$Pr(Error) = P(\Pi_1|\rho_2)P(\rho_2) + P(\Pi_2|\rho_1)P(\rho_2). \tag{17}$$

A projective measurement on a single qubit is given by the projectors $\Pi_1 = |v\rangle \langle v|$ and $\Pi_2 = I - |v\rangle \langle v|$. We are looking for the $|v\rangle$ that minimizes the probability of error. Plugging into the above expression for the probability of error, we have

$$Pr(Error) = \frac{1}{2}\mathrm{tr}(\Pi_1\rho_2) + \frac{1}{2}\mathrm{tr}(\Pi_2\rho_1) \tag{18}$$

$$= \frac{1}{2}(\langle v| \rho_2 |v\rangle + \mathrm{tr}\rho_1 - \langle v| \rho_1 |v\rangle \tag{19}$$

$$= \frac{1}{2}(1 - \langle v| (\rho_1 - \rho_2) |v\rangle). \tag{20}$$

We can see that this will reach its minimum when $|v\rangle$ is the eigenvector of $\rho_1 - \rho_2$ corresponding to the largest eigenvalue. We can see by inspection that such a choice would satisfy the checks given in the hint.

5b: To generalize to a POVM, we can note that $\Pi_2 = I - \Pi_1$ and repeat the construction:

$$Pr(Error) = \frac{1}{2}\mathrm{tr}(\Pi_1\rho_2) + \frac{1}{2}\mathrm{tr}(\Pi_2\rho_1) \tag{21}$$

$$= \frac{1}{2}(1 - \mathrm{tr}\Pi_1(\rho_1 - \rho_2)). \tag{22}$$

Let $|a_i\rangle$ be the eigenvectors of $\rho_1 - \rho_2$ associated with eigenvalues a_i, where I will assume (without loss of generality) that $a_1 \geq a_2$. I claim that $a_1 \geq 0$

and $a_2 \leq 0$, that is, that $\rho_1 - \rho_2$ has one positive and one negative eigenvalue. Both are identically 0 if and only if $\rho_1 = \rho_2$. This is a consequence of both ρ_1 and ρ_2 being positive semidefinite with trace of 1.

We can write the trace in the above expression for the probability of error as

$$\begin{aligned} \mathrm{tr}\Pi_1(\rho_1 - \rho_2) &= \langle a_1 | \, \Pi_1(\rho_1 - \rho_2) \, | a_1 \rangle + \langle a_2 | \, \Pi_1(\rho_1 - \rho_2) \, | a_2 \rangle \qquad (23) \\ &= a_1 \langle a_1 | \, \Pi_1 \, | a_1 \rangle + a_2 \langle a_2 | \, \Pi_1 \, | a_2 \rangle \, . \qquad (24) \end{aligned}$$

Since Π_1 is positive, and a_2 is negative, we want to choose Π_1 such that $\langle a_2 | \, \Pi_1 \, | a_2 \rangle = 0$. We also want $\langle a_1 | \, \Pi_1 \, | a_1 \rangle$ as large as possible. Thus $\Pi_1 = | a_1 \rangle \langle a_1 |$. But this is just the same projective measurement we found in part (a). Thus, we conclude that allowing a general POVM will not improve upon the probability of error we found with a projective measurement.