18.435/2.111 Homework 8 Solutions

1: The ground state of the quantum harmonic oscillator is given by $\psi_0(x) = c_0 e^{-\alpha x^2/2}$, where $\alpha = m\omega/\hbar$ and c_0 is an unimportant constant. If we apply the annihilation operator $a = \sqrt{\frac{1}{2m\omega\hbar}}(m\omega x + ip) = \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x})$ we get

$$\begin{aligned} a\psi_0(x) &= \sqrt{\frac{m\omega}{2\hbar}}(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x})c_0 e^{-\alpha x^2/2} && (1)\\ &= c_0\sqrt{\frac{m\omega}{2\hbar}}(x - \frac{\alpha\hbar x}{m\omega})e^{-\alpha x^2/2} && (2)\\ &= c_0\sqrt{\frac{m\omega}{2\hbar}}(x - x)e^{-\alpha x^2/2} && (3)\\ &= 0 && (4) \end{aligned}$$

2a: First we should note that all of the Pauli matrices are Hermitian, so we can drop all of the †'s from the expression. Next remember that $\sigma_x\sigma_z = -i\sigma_y$, $\sigma_x\sigma_y = i\sigma_z$, and $\sigma_z\sigma_y = -i\sigma_x$. From these we can see the following:

$$\begin{aligned} \sigma_x\tau\sigma_x &= \frac{1}{4}(\sigma_x\rho\sigma_x + \sigma_x^2\rho\sigma_x^2 + \sigma_x\sigma_y\rho\sigma_y\sigma_x + \sigma_x\sigma_z\rho\sigma_z\sigma_x) && (5)\\ &= \frac{1}{4}(\sigma_x\rho\sigma_x + \rho + (i\sigma_z)\rho(-i\sigma_z) + (-i\sigma_y)\rho(i\sigma_y)) && (6)\\ &= \tau, && (7)\\ \sigma_z\tau\sigma_z &= \frac{1}{4}(\sigma_z\rho\sigma_z + \sigma_z\sigma_x\rho\sigma_x\sigma_z + \sigma_z\sigma_y\rho\sigma_y\sigma_z + \sigma_z^2\rho\sigma_z^2) && (8)\\ &= \frac{1}{4}(\sigma_z\rho\sigma_z + (i\sigma_y)\rho(-i\sigma_y) + (-i\sigma_x)\rho(i\sigma_x) + \rho) && (9)\\ &= \tau. && (10) \end{aligned}$$

2b: A very nice way to see this is by remembering that the Pauli matrices together with the identity I form a basis for 2×2 matrices. Thus we can write $\tau = a_0 I + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z$. From this we get

$$\begin{aligned} \sigma_x\tau\sigma_x &= a_0 I + a_1\sigma_x - a_2\sigma_y - a_3\sigma_z && (11)\\ \sigma_z\tau\sigma_z &= a_0 I - a_1\sigma_x - a_2\sigma_y + a_3\sigma_z && (12)\\ \sigma_z\sigma_x\tau\sigma_x\sigma_z &= a_0 I - a_1\sigma_x + a_2\sigma_y - a_3\sigma_z. && (13) \end{aligned}$$

Since all three of these must be equal, we can quickly see that $a_1 = a_2 = a_3 = 0$ and $\tau = a_0 I$. Thus the only trace 1 τ is $\tau = \frac{1}{2}I$.

2c: There are many solutions to this problem, but I will give an intuitive solution that meets the 'scaled unitary' criterion. (This was not part of the problem, though it was originally intended to be.) What we are looking for are a generalization of the Pauli matrices to qutrits. We saw in an earlier problem set one such generalization, using the two matrices

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, \tag{14}$$

where $\omega = e^{2\pi i/3}$. These are the generalizations of σ_x and σ_z to three dimensions. I claim that the set $\{R^a T^b/\sqrt{3}\}$ for $a, b \in \{0, 1, 2\}$ completely randomize a qutrit.

To show this, let's consider the action of these 9 matrices on operators $|c\rangle\langle d|$, where $c, d \in \{0, 1, 2\}$. While these are not density matrices themselves, it should be quickly clear that they form a basis for all density matrices. What is the mapping?

$$\begin{aligned}
|c\rangle\langle d| &\mapsto \sum_{a,b} R^a T^b |c\rangle\langle d| T^{b\dagger} R^{a\dagger}/3 && (15)\\
&= \sum_{a,b} R^a |b + c \bmod 3\rangle\langle d + b \bmod 3| R^{a\dagger}/3 && (16)\\
&= \sum_{a,b} \omega^{a(c-d)} |b + c \bmod 3\rangle\langle b + d \bmod 3|/3 && (17)\\
&= \sum_{b} |b + c \bmod 3\rangle\langle b + c \bmod 3|/3 && (18)\\
&= I/3. && (19)
\end{aligned}$$

We go from (17) to (18) by noting that $\sum_a \omega^{a(c-d)} = \delta_{cd}$. We can now write down the desired result:

$$\begin{aligned}
\rho &= \sum_{c,d} \rho_{cd} |c\rangle\langle d| && (20)\\
&\mapsto \sum_{c} \rho_{cc} I/3 && (21)\\
&= I/3, && (22)
\end{aligned}$$

since $\sum_c \rho_{cc} = \text{tr}\rho = 1$. Thus, these nine matrices completely randomize a qubit.

3a: Let $|\psi\rangle$ be an eigenvector of $U_2^\dagger U_1$ with eigenvalue $e^{i\theta}$. We know that this eigenvector exists as $U_2^\dagger U_1$ is unitary. Then for $\rho = |\psi\rangle\langle\psi|$,

$$\begin{aligned} pU_1\rho U_1^\dagger + (1-p)U_2|\psi\rangle\langle\psi|U_2^\dagger &= U_2(pU_2^\dagger U_1|\psi\rangle\langle\psi|U_1^\dagger U_2 + (1-p)|\psi\rangle\langle\psi|)U_2^\dagger \\ &= U_2(pe^{i(\theta-\theta)}|\psi\rangle\langle\psi|U_1^\dagger U_2 + (1-p)|\psi\rangle\langle\psi|)U_2^\dagger \\ &= U_2|\psi\rangle\langle\psi|U_2^\dagger \qquad (23) \\ &= |\psi'\rangle\langle\psi'| \qquad (24) \end{aligned}$$

where $|\psi'\rangle = U_2|\psi\rangle$. We have shown the existence of a pure input which leads to a pure output.

3b: We need to show that we cannot satisfy the total randomization property if we have fewer than 4 unitaries. It is trivially impossible for only 1 (as any pure input will lead to a pure output), and we showed in (3a) that 2 unitaries are insufficient. We can use the results of (3a) to show that 3 are also insufficient by contradiction. Assume we have U_k and p_k, $k = 1, 2, 3$ such that the randomization property holds. Then we can write

$$\begin{aligned} I/2 &= p_1U_3^\dagger U_1\rho U_1^\dagger U_3 + p_2U_3^\dagger U_2\rho U_2^\dagger U_3 + p_3\rho \qquad (25) \\ &= (1-p_3)(\frac{p_1}{p_1+p_2}U_3^\dagger U_1\rho U_1^\dagger U_3 + \frac{p_2}{p_1+p_2}U_3^\dagger U_2\rho U_2^\dagger U_3) + p_3\rho. \qquad (26) \end{aligned}$$

From part (3a), we know that we can choose a pure input $|\psi\rangle\langle\psi|$ such that the portion in parentheses is a pure state $|\psi'\rangle\langle\psi'|$. Using this we have

$$I/2 = (1-p_3)|\psi'\rangle\langle\psi'| + p_3|\psi\rangle\langle\psi|. \qquad (27)$$

This can only be satisfied if $p_3 = 1/2$. (This is a necessary but not sufficient condition for the equation to hold.) But notice that we could repeat this same procedure to conclude that $p_1 = 1/2$ and $p_2 = 1/2$. This is clearly a contradiction as they need to sum to 1.

Now we want to show that for $m = 4$, $p_k = 1/4$. We can again use (3a). Let's rewrite the combination as

$$(p_1+p_2)(\frac{p_1}{p_1+p_2}U_1\rho U_1^\dagger + \frac{p_2}{p_1+p_2}U_2\rho U_2^\dagger) + p_3U_3\rho U_3^\dagger + p_4U_4\rho U_4^\dagger. \qquad (28)$$

We may again choose the input ψ that leaves the item in parentheses pure, from which we see that

$$(p_1+p_2)|\psi'\rangle\langle\psi'| + p_3U_3\rho U_3^\dagger + p_4U_4\rho U_4^\dagger = I/2. \qquad (29)$$

We can conclude from this that $p_1 + p_2 \leq 1/2$. To see this, bracket both sides with $|\psi'\rangle$:

$$\begin{aligned}\langle\psi'| [(p_1+p_2)|\psi'\rangle\langle\psi'| + p_3 U_3 \rho U_3^\dagger + U_4 \rho U_4^\dagger] |\psi'\rangle &= \langle\psi'| I/2 |\psi'\rangle \quad (30)\\ p_1 + p_2 + p_3 \langle\psi'| U_3 \rho U_3^\dagger |\psi'\rangle + p_4 \langle\psi'| U_4 \rho U_4^\dagger |\psi'\rangle &= 1/2 \quad (31)\\ p_1 + p_2 &\leq 1/2 \quad (32)\end{aligned}$$

The last line arises because $U_k \rho U_k^\dagger$ is a positive semidefinite matrix, and thus $\langle\psi'| U_k \rho U_k^\dagger |\psi'\rangle \geq 0$ for all $|\psi'\rangle$.

Notice, there was nothing special about the choice of p_1 and p_2. We can use a similar argument to conclude the $p_k + p_{k'} \leq 1/2$ for any $k \neq k'$. Since we still have to sum to one, we can conclude that $p_k = 1/4$.

4: There are several ways to approach this problem. I'm going to follow an approach by first considering generically how to interpret a quantum operation by Kraus operators $\{A_i\}$ on a pure state $|\psi\rangle$. We may interpret the outcome as being in state $A_i|\psi\rangle / \|A_i|\psi\rangle\|$ with probability $\|A_i|\psi\rangle\|^2$. If you write out the density matrix corresponding to this, you will see that it becomes $\sum_i A_i|\psi\rangle\langle\psi| A_i^\dagger$, which is the relationship we want.

Now to the problem at hand. We want the pure state input $|\psi\rangle$ such that the output is also pure. Using the above interpretation, we conclude that $A_i|\psi\rangle$ must point in the same direction for every i. Thus, $A_1|\psi\rangle \propto A_2|\psi\rangle \propto A_3|\psi\rangle$. We can come closer to a solution by noting that A_3 is non-singular in this case, so we can instead solve $A_3^{-1}A_1|\psi\rangle \propto A_3^{-1}A_2|\psi\rangle \propto |\psi\rangle$. In other words, we want the joint eigenvectors of $A_3^{-1}A_1$ and $A_3^{-1}A_2$. This is now just a numerical problem. Plugging in the numbers and solving, we see that the two pure states are $\begin{bmatrix} \sqrt{2/3} \\ \pm\sqrt{1/3} \end{bmatrix}$.