18.435/2.111 Homework # 8

Due Thursday, November 16

1: Show that for the harmonic oscillator, $a\,|\,0\rangle = 0$ by explicitly applying the lowering (annhilation) operator a to the ground state wave function $|\,0\rangle$ as computed in class.

2: Prove the formula $\tau = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ where ρ is an arbitrary density matrix for a qubit and

$$\tau = \frac{1}{4}\left(\rho + \sigma_x \rho \sigma_x^\dagger + \sigma_y \rho \sigma_y^\dagger + \sigma_z \rho \sigma_z^\dagger\right)$$

as follows:

2a: Show that

$$\tau = \sigma_x \tau \sigma_x^\dagger = \sigma_z \tau \sigma_z^\dagger$$

2b: Show that any trace 1 matrix τ satisfying

$$\tau = \sigma_x \tau \sigma_x^\dagger = \sigma_z \tau \sigma_z^\dagger$$

must be $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2c: Give a set of nine 3×3 matrices which completely randomize a qutrit (3-dimensional quantum state).

3a: Suppose you have a quantum operation of the following form acting on a qubit

$$\rho \rightarrow p\, U_1 \rho U_1^\dagger + (1-p) U_2 \rho U_2^\dagger$$

where $0 < p < 1$ is a probability. (This operation is a probabilistic mixture of two unitaries.) Show that there is at least one pure quantum state which is taken to a pure quantum state by this operation.

3b: Suppose that you have a quantum operation that is a probabilistic mixture of unitaries

$$\rho \rightarrow \sum_{k=1}^{m} p_k\, U_k \rho U_k^\dagger, \quad \text{where } \sum_{k=1}^{m} p_k = 1$$

which takes all density matrices to the matrix $\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that there are at least four unitaries in the mixture, and that if there are exactly four, then the p_k are all $\frac{1}{4}$. (Note: I labeled these problems 3a and 3b because 3a can be useful in proving 3b, although there are other good ways of proving this as well.)

4: Consider the quantum operation given in Krauss operator sum notation

$$\rho \to \sum A_i \rho A_i^\dagger$$

by three Krauss operators:

$$A_1 = \frac{\sqrt{5}}{2\sqrt{77}} \begin{pmatrix} 3 & -3\sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}, \qquad A_2 = \frac{\sqrt{5}}{2\sqrt{77}} \begin{pmatrix} 3 & 3\sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix}, \qquad A_3 = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

There are two distinct pure states which are taken to pure states by this quantum operation. What are they?