18.435/2.111 Homework 7 Solutions

1: I'll sketch the procedure. Access the database at random and achieve some *kety*. Design an oracle such that $O|x\rangle|y\rangle = -|x\rangle|y\rangle$ if $x < y$ and is identity otherwise. We can now do Grover's search and find $y' < y$ in $O(\sqrt{N})$ steps. Note that this can be done using the randomized procedure sketched in class for when we don't know how many solutions are marked. As each iteration works with probability 1/2, we repeat if $y' \geq y$ and we replace y with y' if $y' < y$. We can continue this procedure until we feel confident (*i.e.* probability greater than 1/2) that we have found the minimum.

How long does this take? Intuition tells us that on average, each time we find a $y' < y$ we have divided our search space in half. This would suggest that we find a smaller y' $O(\log(N))$ times, each time at a cost of $O(\sqrt{N})$. Thus, we access the database $O(\log(N)\sqrt{N})$ times.

As some of you discovered, this procedure is discussed in detail in a paper available at arxiv.org: quant-ph/9607014 by Dürr and Høyer. They, in fact, show that you can get the minimum with probability greater than 1/2 in only $O(\sqrt{N})$ database accesses. Anyone interested in the details can read the short paper there. Notice that they are counting time intervals, not database accesses as our problem asks, which should help you interpret their results in our context.

2: Let's first note that we can write each of the POVM elements as rank 1 operators $|a_i\rangle\langle a_i|$ where the $|a_i\rangle$ are not necessarily normalized. We can see that $|a_1\rangle = [1/\sqrt{2} \quad 1/2\sqrt{2}]^T$, $|a_2\rangle = [1/\sqrt{2} \quad -1/2\sqrt{2}]^T$, and $|a_3\rangle = [0 \quad \sqrt{3}/2]^T$.

The next step will be to find 3 rank 1 projectors $\Pi_i = |c_i\rangle\langle c_i|$ on 2 qubits such that $\langle\psi|\langle 0|\Pi_i|\psi\rangle|0\rangle = \langle\psi|a_i\rangle\langle a_i|\psi\rangle$. We can accomplish this by writing $|c_i\rangle = |a_i\rangle|0\rangle + |b_i\rangle|1\rangle$ and choosing an appropriate $|b_i\rangle$ so that $|c_i\rangle$ have unit length and are orthogonal. As it turns out, we can choose $|b_1\rangle = \sqrt{3/8}|0\rangle$, $|b_2\rangle = -\sqrt{3/8}|0\rangle$ and $|b_3\rangle = 1/2|0\rangle$. We should also note that we should define a fourth projector $\Pi_4 = I - \Pi_1 - \Pi_2 - \Pi_3$ for completeness.

There is one final step: instead of projecting onto a specified basis, we want to project onto the computational basis on two qubits. What unitary rotation do we need to make? We need to rotate each of our $|c_i\rangle$ onto the set $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Define

$$U = |00\rangle\langle c_1| + |01\rangle\langle c_2| + |10\rangle\langle c_3| + |11\rangle\langle c_4|.$$

This is the unitary rotation that we want. Shall we fill in some numbers?

$$|c_1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ \sqrt{3/8} \\ 1/2\sqrt{2} \\ 0 \end{bmatrix} \quad |c_2\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -\sqrt{3/8} \\ -1/2\sqrt{2} \\ 0 \end{bmatrix} \quad |c_3\rangle = \begin{bmatrix} 0 \\ -1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix} \quad |c_4\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 1/\sqrt{2} & \sqrt{3/8} & 1/2\sqrt{2} & 0 \\ 1/\sqrt{2} & -\sqrt{3/8} & -1/2\sqrt{2} & 0 \\ 0 & -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Notice that this is not a unique solution - we had some freedom in choosing $|b_i\rangle$, and we could just as easily have mapped the $|c_i\rangle$ to different computational bases.