18.435/2.111 Homework # 4

Due Thursday, October 12

1: Recall that the matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

changes the $|\, z\pm\rangle$ basis to $|\, x\pm\rangle$ and vice versa. Find the matrix which changes the $|\, x\pm\rangle$ basis to the $|\, y\pm\rangle$ basis, as well as the matrix that changes the $|\, z\pm\rangle$ basis to the $|\, y\pm\rangle$ basis. Use this to change the basis for the CNOT gate to the $|\, y\pm\rangle$ basis.

2: Suppose we have a POVM with n rank 1 elements, $|\, e_1\rangle\langle e_1\,|$, $|\, e_2\rangle\langle e_2\,|$, ..., $|\, e_n\rangle\langle e_n\,|$ operating in an n-dimensional vector space (remember that $|\, e_i\rangle$ is not necessarily normalized). Prove that this POVM is actually a projective measurement. One way to proceed is first to use the trace to show that the $|\, e_i\rangle$ must be unit vectors. Then, show that if the $|\, e_i\rangle$ are not orthogonal, there is some quantum state $|\, v\rangle$ such that the probabilities of all the outcomes of this measurement on $|\, v\rangle$ add up to more than 1.

3: Suppose we have a POVM with elements E_1, ..., E_m, with element E_i associated with a real number r_i. We can define the expected value of this POVM on a state $|\, \psi\rangle$ as

$$\sum_i r_i \,\langle \psi \,|\, E_i \,|\, \psi\rangle$$

3a: Show that there is a Hermitian observable whose expected value is the same on all states $|\, \psi\rangle$ as the expected value of the above POVM.
3b: Is this Hermitian matrix unique?
3c: Give an example where the probability distributions of the outcomes r_i for the above POVM cannot be duplicated by an Hermitian matrix, even though the expectated values can be.

4: Taken from Nielsen and Chuang, Problem 2.70.
4a: Suppose that Alice and Bob share one of the four Bell states

$$\frac{1}{\sqrt{2}} \left(|\, 00\rangle_{AB} \pm |\, 11\rangle_{AB}\right) \qquad \frac{1}{\sqrt{2}} \left(|\, 01\rangle_{AB} \pm |\, 10\rangle_{AB}\right)$$

Show that for any observable M, on Alice's qubit, the expected value of $M \otimes I$ is the same for all four Bell states.
4b: Suppose Alice and Bob are doing superdense coding, and an eavesdropper Eve intercepts the qubit Alice is sending to Bob. She has clearly prevented the information from arriving at Bob. But can Eve discover anything about the message being transmitted? If so, how? If not, why not?