18.435/2.111 Homework 2 Solutions

1: Let's begin with $|x+\rangle$. Both the probability and the residual state can be obtained by computing the projection of the GHZ state onto $|x+\rangle$ in the first qubit.

$$\begin{aligned} |\psi\rangle_{23} &= {}_1\langle x+|\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle) \\ &= \frac{1}{2}({}_1\langle 0|+{}_1\langle 1|)(|000\rangle+|111\rangle) \\ &= \frac{1}{2}(|00\rangle+|11\rangle) \end{aligned}$$

(Notice that there is an implied identity operator in the above equation. The desired projection could be written as $\langle x+|\otimes I\otimes I$ to indicate $\langle x+|$ on just the first qubit. This is usually dropped for convenience. To avoid confusion, it is often acceptable simply to label the kets with subscripts.) The probability of measuring $|x+\rangle$ is then $p(x+)=|\langle\psi|\psi\rangle|=1/2$ and the residual state of the second and third qubits is the renormalization of $|\psi\rangle$ to unit length:

$$\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle). \tag{1}$$

We can calculate $|x-\rangle$ in the same way:

$$\begin{aligned} |\psi\rangle_{23} &= {}_1\langle x-|\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle) \\ &= \frac{1}{2}({}_1\langle 0|-{}_1\langle 1|)(|000\rangle+|111\rangle) \\ &= \frac{1}{2}(|00\rangle-|11\rangle). \end{aligned}$$

From this we see that $p(x-)=1/2$ and the residual state is

$$\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle). \tag{2}$$

2: The probability of the GHZ state being measured in the state $|\psi\rangle$ is given by

$$p(\psi)=|\langle\psi|\frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)|^2. \tag{3}$$

We can see from this equation that we need to compute the two inner products $\langle\psi|000\rangle$ and $\langle\psi|111\rangle$. With a little thought, we can see that

$$\langle x\pm x\pm x\pm|GHZ\rangle=\frac{1}{4}(\langle 000|000\rangle\pm\langle 111|111\rangle) \tag{4}$$

where the $\pm$ will be + for an odd number of $x+$ and $-$ for an even number of $x+$. Thus, the probability of measuring and even number of $x+$ is 0.

For the first qubit in $x\pm$ and the second and third in $y\pm$, let's write out the inner products for each of the even number of + states:

$$\begin{aligned}
\langle x+, y+, y-|GHZ\rangle &= \frac{1}{4}(\langle 000|000\rangle + (+1)(i)(-i)\langle 111|111\rangle) = 1/2 \\
\langle x+, y-, y+|GHZ\rangle &= \frac{1}{4}(\langle 000|000\rangle + (+1)(-i)(i)\langle 111|111\rangle) = 1/2 \\
\langle x-, y+, y+|GHZ\rangle &= \frac{1}{4}(\langle 000|000\rangle + (-1)(i)(i)\langle 111|111\rangle) = 1/2 \\
\langle x-, y-, y-|GHZ\rangle &= \frac{1}{4}(\langle 000|000\rangle + (-1)(-i)(-i)\langle 111|111\rangle) = 1/2.
\end{aligned}$$

Squaring these and summing, we see that the probability of getting an even number of + states is 1.

3a: Let's assume the opposite. That is assume $A_1 = 1, A_2 = A_3 = A_4 = -1$. We can see that this is a contradiction by noting that $A_1A_2A_3A_4 = -1$, but when we multiply it out in terms for $f_i(\cdot)$, we get

$$A_1A_2A_3A_4 = f_1(x)^2 f_1(y)^2 f_2(x)^2 f_2(y)^2 f_3(x)^2 f_3(y)^2 = 1. \tag{5}$$

Thus, either $A_1 = -1$ or one of A_2, A_3, or A_4 is -1, which is the desired result.

3b: This problem was made more confusing by an inadvertent choice in problem 2. While the same arguments hold as the problem was given, it is significantly easier to see if the state of interest were $\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$. If we were to redo problem 2 for this state, we would determine that the probability of an even number of + states when measuring in the $(x\pm, x\pm, x\pm)$ basis is 1, and the probability of an even number of + states when measuring in the $(x\pm, y\pm, y\pm)$ is 0.

We then equate measuring the i^{th} qubit to $f_i(\cdot)$ and the argument indicates whether we are measuring in the x or y direction. The two cases from 2 now correspond to A_1 and A_2. Further, it should be clear given a little thought that A_3 and A_4 have the same behavior as A_2 when interpreted this way. But here we see a contradiction! Now A_1 is always positive, and A_2, A_3, and A_4 are always negative, which contradicts what we proved in 3a. This is an indication of fundamentally non-classical behavior of the measurement of an entangled state.

4:

$$J_xJ_y = \frac{1}{4}(\sigma_x \otimes id + id \otimes \sigma_x)(\sigma_y \otimes id + id \otimes \sigma_y) \tag{6}$$
$$= \frac{1}{4}(\sigma_x\sigma_y \otimes id + \sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x + id \otimes \sigma_x\sigma_y) \tag{7}$$
$$= \frac{1}{4}(i\sigma_z \otimes id + i(id) \otimes \sigma_z + \sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x) \tag{8}$$

$$J_yJ_x = \frac{1}{4}(\sigma_y \otimes id + id \otimes \sigma_y)(\sigma_x \otimes id + id \otimes \sigma_x) \tag{9}$$
$$= \frac{1}{4}(\sigma_y\sigma_x \otimes id + \sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x + id \otimes \sigma_y\sigma_x) \tag{10}$$
$$= \frac{1}{4}(-i\sigma_z \otimes id - i(id) \otimes \sigma_z + \sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x) \tag{11}$$

We have used the fact that $\sigma_x\sigma_y = i\sigma_z$ and $\sigma_y\sigma_x = -i\sigma_z$. We can now write the commutator as

$$J_xJ_y - J_yJ_x = \frac{1}{4}(2i\sigma_z \otimes id + 2i(id) \otimes \sigma_z) \tag{12}$$
$$= iJ_z. \tag{13}$$

We could continue with the computations through similar brute force algebra (and you will receive full credit if you did), but it is more useful to introduce a commutator identity that comes in handy:

$$[A, BC] = B[A, C] + [A, B]C \tag{14}$$

Furthermore, from the exact same calculations that we did with $[J_x, J_y]$, we compute that $[J_x, J_z] = -iJ_y$. We can use these facts to compute the desired commutators:

$$J_xJ_y^2 - J_y^2J_x = [J_x, J_y^2] \tag{15}$$
$$= J_y[J_x, J_y] + [J_x, J_y]J_y \tag{16}$$
$$= iJ_yJ_z + iJ_zJ_y \tag{17}$$

$$J_xJ_z^2 - J_z^2J_x = [J_x, J_z^2] \tag{18}$$
$$= J_z[J_x, J_z] + [J_x, J_z]J_z \tag{19}$$
$$= -iJ_zJ_y - iJ_yJ_z \tag{20}$$

From these we can see that

$$[J_x, J_x^2 + J_y^2 + J_z^2] = iJ_yJ_z + iJ_zJ_y - iJ_zJ_y - iJ_yJ_z = 0, \quad (21)$$

which was stated in class.

5: First let's calculate RT and TR:

$$RT = \begin{pmatrix} 0 & 0 & 1 \\ e^{2\pi i/3} & 0 & 0 \\ 0 & e^{4\pi i/3} & 0 \end{pmatrix} \quad (22)$$

$$TR = \begin{pmatrix} 0 & 0 & e^{4\pi i/3} \\ 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \end{pmatrix} = e^{4\pi i/3} RT \quad (23)$$

From this observation, we can state several others that will be of value in the next section:

$$\begin{aligned} RT &= e^{2\pi i/3} TR & (24) \\ R^a T &= e^{2a\pi i/3} TR^a & (25) \\ R^a T^b &= e^{2ab\pi i/3} T^b R^a. & (26) \end{aligned}$$

Also, we should note that R and T are both unitary, so $R^{-1} = R^\dagger$ and $T^{-1} = T^\dagger$. Also note the $T^3 = R^3 = I$.

To show orthogonality of the given states, let's write the inner product between any two of them. We'll use the parameters a, b, a', and b'.

$$\begin{aligned} &\frac{1}{3}(\langle 00| + \langle 11| + \langle 22|)(R^{a'}T^{b'} \otimes I)^\dagger (R^a T^b \otimes I)(|00\rangle + |11\rangle + |22\rangle) \\ &= \frac{1}{3}(\langle 00| + \langle 11| + \langle 22|)T^{-b'} R^{a-a'} T^b \otimes I(|00\rangle + |11\rangle + |22\rangle) \quad (27) \\ &= \frac{e^{2b(a-a')\pi i/3}}{3}(\langle 00| + \langle 11| + \langle 22|)T^{b-b'} R^{a-a'} \otimes I(|00\rangle + |11\rangle + |22\rangle) \end{aligned}$$

Let's now apply the R and T operators to the kets. This is easier done by understanding the qualitative behavior of each. Note that R applies the phase $e^{2\pi i/3}$ to $|1\rangle$ and $e^{4\pi i/3}$ to $|2\rangle$. Note that T adds 1, modulo 3, to a ket: $T|i\rangle = |(i+1)\mathrm{mod}3\rangle$. Using this, we can write the above inner product as

$$\frac{e^{2b(a-a')\pi i/3}}{3}(\langle 00| + \langle 11| + \langle 22|)(|b-b'\rangle |0\rangle + e^{2\pi(a-a')i/3} |1+b-b'\rangle |1\rangle + e^{4\pi(a-a')i/3} |2+b-b'\rangle |2\rangle) \quad (28)$$

Let's call this quantity X. For $a = a'$ and $b = b'$, $X = 1$, which shows that the states are of unit length. For $b \neq b'$, we can easily see that all of the kets on the right are orthogonal to the bras on the left. For $b = b'$ and $a \neq a'$,

$$X \propto 1 + e^{2\pi(a-a')i/3} + e^{4\pi(a-a')i/3} = 1 + e^{2\pi i/3} + e^{4\pi i/3} = 0. \qquad (29)$$

This proves the orthonormality of the states.

6: We have already done the heavy lifting for the superdense coding in the orthogonality proof of problem 5. Alice and Bob begin with the entangled qutrit state

$$\frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle) \qquad (30)$$

which we saw in the last problem. Alice can then send one of the nine messages encoded in $\{a, b\}$ by applying $R^a T^b$ to her qutrit, and sending it to Bob. We have already shown that these 9 states form an orthonormal basis, which means that Bob can perfectly distinguish between them. He thus receives the classical message with only the transmission of a single qutrit.