18.435/2.111 Homework # 2

Due Thursday, September 28

Let us recall the definitions

$$\begin{aligned} |\,z+\rangle &= |\,0\rangle \\ |\,z-\rangle &= |\,1\rangle \\ |\,x+\rangle &= \frac{1}{\sqrt{2}}(|\,0\rangle + |\,1\rangle) \\ |\,x-\rangle &= \frac{1}{\sqrt{2}}(|\,0\rangle - |\,1\rangle) \\ |\,y+\rangle &= \frac{1}{\sqrt{2}}(|\,0\rangle + i\,|\,1\rangle) \\ |\,y-\rangle &= \frac{1}{\sqrt{2}}(|\,0\rangle - i\,|\,1\rangle) \end{aligned}$$

1: Suppose you have three qubits in the GHZ state

$$\frac{1}{\sqrt{2}}(|\,000\rangle + |\,111\rangle)$$

and you measure the first one in the $|\,x\pm\rangle$ basis. What are the probabilities for each measurement result, and what is the residual state on the second and third qubits in each case?

2: Suppose we have the GHZ state above, and we measure all three qubits in the $|\,x\pm\rangle$ basis. What is the probability we obtain an even number of + states in the result? Suppose we measure one in the $|\,x\pm\rangle$ basis and the other two in the $|\,y\pm\rangle$ basis. What is the probability we obtain an even number of + states in the result?

3a: (Note that this problem has no quantum mechanics in it.) Suppose f_1, f_2, and f_3 are functions mapping the set $\{x, y\}$ to the set $\{1, -1\}$. Define

$$\begin{aligned} A_1 &= f_1(x)f_2(x)f_3(x), \qquad & A_2 &= f_1(x)f_2(y)f_3(y) \\ A_3 &= f_1(y)f_2(x)f_3(y), \qquad & A_4 &= f_1(y)f_2(y)f_3(x). \end{aligned}$$

Show that either $A_1 = -1$ or one of A_2, A_3, A_4 eqnals $+1$.

3b: What is the relationship between problem 2 and problem 3a?

More problems on reverse.

4: Suppose we have two qubits. Consider the observables

$$\begin{aligned} J_x &= \frac{1}{2}(\sigma_x \otimes id + id \otimes \sigma_x) \\ J_y &= \frac{1}{2}(\sigma_y \otimes id + id \otimes \sigma_y) \\ J_z &= \frac{1}{2}(\sigma_z \otimes id + id \otimes \sigma_z) \end{aligned}$$

Compute

$$J_x J_y - J_y J_x.$$

Can you express the result in terms of J_z?
Compute

$$J_x J_y^2 - J_y^2 J_x,$$

and

$$J_x J_z^2 - J_z^2 J_x.$$

The results should show you that J_x commutes with $J_x^2 + J_y^2 + J_z^2$, as stated in class.

5: Consider a qutrit, i.e., a three-state quantum system with basis states $|0\rangle$, $|1\rangle$, $|2\rangle$. Let

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}$$

and

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Find the relationship between RT and TR. Show that the nine states

$$\frac{1}{\sqrt{3}}(R^a T^b \otimes I)(|00\rangle + |11\rangle + |22\rangle)$$

with $0 \le a, b \le 2$ are all orthogonal.

6: Consider a qutrit, i.e., a three-state quantum system with basis states $|0\rangle$, $|1\rangle$, $|2\rangle$. Show how you can use superdense coding to transmit nine distinct possibilities using one qutrit.