18.435/2.111 Homework 10 Solutions

1: The Pauli matrices, together with the identity, form a basis for all qubit operations. From this, we can quickly see that the operator elements E_0 and E_1 can be represented as $E_0 = \frac{1+\sqrt{1-\gamma}}{2}I + \frac{1-\sqrt{1-\gamma}}{2}\sigma_z$ and $E_1 = \sqrt{\gamma}/2\sigma_x + i\sqrt{\gamma}/2\sigma_y$. The intuitive answer, which turns out to be true, is that the probability of measuring each of these errors is the magnitude squared of each coefficient. Thus $P(\text{No Error}) = \frac{(1+\sqrt{1-\gamma})^2}{4}$, $P(\sigma_z) = \frac{(1-\sqrt{1-\gamma})^2}{4}$, $P(\sigma_x) = \frac{\gamma}{4}$, and $P(\sigma_y) = \frac{\gamma}{4}$.

I've said this is the right answer, now I'd like to justify this. For a code that can correct for one error, that implies that the subspaces corresponding to a pauli error on the second qubit are all orthogonal. Let's define P_C as the projection operator onto the code subspace. The subspace of a pauli error σ_i on the second qubit is then $\sigma_i P_C \sigma_i$ (remembering that the paulis are Hermitian, so I've dropped the †). I should probably carry along a subscript 2 to indicate that the action is on the second qubit, but this will be cumbersome, so please just understand all of the operations for the rest of this problem to be on the second qubit, with identity on the rest. The syndrome measurement then projects onto each of these subspaces. Let's do the case of projecting onto the $\sigma_z P_C \sigma_z$ subspace. Let's call the input state ρ, and remember that the input lies on the codespace, so $\rho = P_C \rho P_C$. Let's call the channel (i.e. the amplitude damping error on the second qubit) as $\mathcal{E}$.

The probability of measuring a σ_z error is then

$$\begin{aligned} \mathrm{tr}\sigma_z P_C \sigma_z \mathcal{E}(\rho) &= \sum_{k=1}^{2} \mathrm{tr}\sigma_z P_C \sigma_z E_k \rho E_k^\dagger & (1)\\ &= \mathrm{tr}\sigma_z P_C \sigma_z (\alpha_0 I + \alpha_z \sigma_z) P_C \rho P_C (\alpha_0^* I + \alpha_z^* \sigma_z) & (2)\\ &+ \mathrm{tr}\sigma_z P_C \sigma_z (\alpha_x \sigma_x + \alpha_y \sigma_y) P_C \rho P_C (\alpha_x^* \sigma_x + \alpha_y^* \sigma_y). & (3) \end{aligned}$$

Here I have called the coefficients derived above as α_0, α_x, etc. I've also inserted a P_C around the input density ρ. Rather than write out all of the terms, let's look at a few terms from which we will quickly see the pattern develop. First, the α_z term:

$$\mathrm{tr}\sigma_z P_C \sigma_z \alpha_z \sigma_z P_C \rho P_C \alpha_z^* \sigma_z = |\alpha_z|^2 \mathrm{tr} P_C \rho P_c = |\alpha_z|^2. \quad (4)$$

Here we've used the cyclic property of the trace and we've gotten the desired probability. So now we need to show that all of the other terms have to go to zero. Let's look at a single cross term:

$$\mathrm{tr}\sigma_z P_C \sigma_z \alpha_z \sigma_z P_C \rho P_C \alpha_0^* I = 0, \quad (5)$$

since $P_C(\sigma_z P_C \sigma_z) = 0$ is the product of projectors onto orthogonal subspaces. In fact, this same trick causes all of the remaining terms to cancel, and we see that the intuitive result is in fact the right result.

2a:

$$\begin{aligned}
\frac{1}{\sqrt{2}} \langle v_\theta| \langle v_\theta| (|01\rangle - |10\rangle) &= \frac{1}{\sqrt{2}}(\cos\theta\sin\theta - \cos\theta\sin\theta) && (6)\\
&= 0. && (7)
\end{aligned}$$

2b: Let's write out $|v_\theta\rangle |v_\theta\rangle \langle v_\theta| \langle v_\theta|$ as a 4×4 matrix:

$$\begin{bmatrix} \cos^4\theta & \cos^3\theta\sin\theta & \cos^3\theta\sin\theta & \cos^2\theta\sin^2\theta \\ \cos^3\theta\sin\theta & \cos^2\theta\sin^2\theta & \cos^2\theta\sin^2\theta & \cos\theta\sin^3\theta \\ \cos^3\theta\sin\theta & \cos^2\theta\sin^2\theta & \cos^2\theta\sin^2\theta & \cos\theta\sin^3\theta \\ \cos^2\theta\sin^2\theta & \cos\theta\sin^3\theta & \cos\theta\sin^3\theta & \sin^4\theta \end{bmatrix} \tag{8}$$

From this, we see that we need to evaluate 5 different integrals, given below:

$$\begin{aligned}
\frac{1}{\pi}\int_{\theta=0}^{\pi} \cos^4\theta d\theta &= \frac{3}{8} && (9)\\
\frac{1}{\pi}\int_{\theta=0}^{\pi} \cos^3\theta\sin^1\theta d\theta &= 0 && (10)\\
\frac{1}{\pi}\int_{\theta=0}^{\pi} \cos^2\theta\sin^2\theta d\theta &= \frac{1}{8} && (11)\\
\frac{1}{\pi}\int_{\theta=0}^{\pi} \cos^1\theta\sin^3\theta d\theta &= 0 && (12)\\
\frac{1}{\pi}\int_{\theta=0}^{\pi} \sin^4\theta d\theta &= \frac{3}{8}. && (13)
\end{aligned}$$

Filling in the numbers, we have the density matrix

$$\frac{1}{8}\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}. \tag{14}$$

Now, note that $\left|v_{\theta+\pi/2}\right\rangle = \cos(\theta+\pi/2)|0\rangle + \sin(\theta+\pi/2)|1\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle$. By a similar set of calculations to those above, we find that $|v_\theta\rangle |v_\theta\rangle \left\langle v_{\theta+\pi/2}\right| \left\langle v_{\theta+\pi/2}\right|$ is given by

$$\begin{bmatrix} \cos^2\theta\sin^2\theta & -\cos^3\theta\sin\theta & -\cos^3\theta\sin\theta & \cos^4\theta \\ \cos\theta\sin^3\theta & -\cos^2\theta\sin^2\theta & -\cos^2\theta\sin^2\theta & \cos^3\theta\sin\theta \\ \cos\theta\sin^3\theta & -\cos^2\theta\sin^2\theta & -\cos^2\theta\sin^2\theta & \cos^3\theta\sin\theta \\ \sin^4\theta & -\cos\theta\sin^3\theta & -\cos\theta\sin^3\theta & \sin^2\theta\cos^2\theta \end{bmatrix}, \tag{15}$$

which when we integrate gives

$$\frac{1}{8}\begin{bmatrix}1 & 0 & 0 & 3\\ 0 & -1 & -1 & 0\\ 0 & -1 & -1 & 0\\ 3 & 0 & 0 & 1\end{bmatrix}. \tag{16}$$

Now we want to write these in terms of the Bell states $|\beta_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\beta_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\beta_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, and $|\beta_4\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. We can see these to be:

$$\begin{aligned}
\frac{1}{\pi}\int_{\theta=0}^{\pi} |v_\theta\rangle |v_\theta\rangle \langle v_\theta| \langle v_\theta| \, d\theta &= \frac{1}{2}|\beta_1\rangle\langle\beta_1| + \frac{1}{4}|\beta_2\rangle\langle\beta_2| + \frac{1}{4}|\beta_3\rangle\langle\beta_3| \\
\frac{1}{\pi}\int_{\theta=0}^{\pi} |v_\theta\rangle |v_\theta\rangle \langle v_{\theta+\pi/2}| \langle v_{\theta+\pi/2}| \, d\theta &= \frac{1}{2}|\beta_1\rangle\langle\beta_1| - \frac{1}{4}|\beta_2\rangle\langle\beta_2| - \frac{1}{4}|\beta_3\rangle\langle\beta_3| .
\end{aligned}$$

2c: Let's write out the desired integral and apply the results from (2b):

$$\begin{aligned}
\frac{1}{\pi}\int_{\theta=0}^{\pi} |e_\theta\rangle\langle e_\theta| \, d\theta &= \frac{1}{\pi}\int_{\theta=0}^{\pi} d\theta(|s|^2 |v_\theta\rangle |v_\theta\rangle \langle v_\theta| \langle v_\theta| + st^* |v_\theta\rangle |v_\theta\rangle \langle v_{\theta+\pi/2}| \langle v_{\theta+\pi/2}| \\
&+ s^*t |v_{\theta+\pi/2}\rangle |v_{\theta+\pi/2}\rangle \langle v_\theta| \langle v_\theta| + |t|^2 |v_{\theta+\pi/2}\rangle |v_{\theta+\pi/2}\rangle \langle v_{\theta+\pi/2}| \langle v_{\theta+\pi/2}|) \\
&= (|s|^2 + |t|^2)(\frac{1}{2}|\beta_1\rangle\langle\beta_1| + \frac{1}{4}|\beta_2\rangle\langle\beta_2| + \frac{1}{4}|\beta_3\rangle\langle\beta_3|) \\
&+ (st^* + s^*t)(\frac{1}{2}|\beta_1\rangle\langle\beta_1| - \frac{1}{4}|\beta_2\rangle\langle\beta_2| - \frac{1}{4}|\beta_3\rangle\langle\beta_3|)
\end{aligned} \tag{17}$$

Since $\{|\beta_i\rangle\}$ form a basis for 2 qubits, we know that $\sum_{i=1}^{4} |\beta_i\rangle\langle\beta_i| = I$, so we want to choose s and t such that the quantity above becomes $\sum_{i=1}^{3} |\beta_i\rangle\langle\beta_i|$. A little algebra shows that we want $|s|^2 + |t|^2 = 3$ and $st^* + s^*t = -1$. One such choice for s and t is $s = \frac{1+\sqrt{2}}{\sqrt{2}}$ and $t = \frac{1-\sqrt{2}}{\sqrt{2}}$.

3a: We have the following state, and we want to choose α in such a way that it is a product state:

$$s|00\rangle - |t|\,|11\rangle + \frac{\alpha}{\sqrt{2}}|01\rangle - \frac{\alpha}{\sqrt{2}}|10\rangle . \tag{18}$$

Notice that s and t are real and have opposite signs. I have written the t as $-|t|$ for convenience as it is easier to see the proper factorization. If this is going to be a product state, it will be of the form

$$(a_1|0\rangle - a_2|1\rangle)(a_3|0\rangle + a_4|1\rangle) \tag{19}$$

where the a_i are real, positive numbers. Furthermore, we can see that $a_1 a_3 = s$, $a_2 a_4 = |t|$, and $a_2 a_3 = a_1 a_4 = \frac{\alpha}{\sqrt{2}}$. We can see that this is satisfied by $a_1 = a_3 = \sqrt{s}$ and $a_2 = a_4 = \sqrt{|t|}$. The final factorization is then

$$(\sqrt{s}\,|0\rangle - \sqrt{|t|}\,|1\rangle)(\sqrt{s}\,|0\rangle + \sqrt{|t|}\,|1\rangle). \tag{20}$$

From this, we can read off the desired values. We compute c by finding the length of the above vector:

$$c^2 = (s + |t|)^2 \tag{21}$$

$$\Rightarrow \quad c = s + |t| = \frac{1+\sqrt{2}}{\sqrt{2}} + \frac{-1+\sqrt{2}}{\sqrt{2}} = 2. \tag{22}$$

We can read the value of α directly from the factorization:

$$\alpha = \sqrt{2}\sqrt{s|t|} = \sqrt{2}\sqrt{\frac{(1+\sqrt{2})}{\sqrt{2}}\frac{(-1+\sqrt{2})}{\sqrt{2}}} = 1. \tag{23}$$

Finally, the inner product (remember to normalize!) is:

$$\langle v_{\theta_1} | v_{\theta_2} \rangle = \frac{s - |t|}{c} = \frac{1}{\sqrt{2}}. \tag{24}$$

3b: This restriction on α arises from the constraint that all elements in a POVM add (or integrate in the continuous case) to I. If we were to allow $\alpha > 1$, then our POVM elements would add up to greater than I, which is not allowed.

3c: The only classical communication that needs to be made is the angle of projection θ_i. Alice and Bob need to communicate this result so that their operations $|\theta_i\rangle\langle\theta_i|$ are always a fixed angle apart, which angle is given by $|\langle\theta_1|\theta_2\rangle|$.

In the case we're considering, we can say a little more. We calculated the inner product in (3a), and from this, we know that the required angle between the measurements is $\pi/4$. The fact that the angle is $\pi/4$ means that $\pi/2$ minus the angle is still $\pi/4$. If Alice and Bob choose bases at $\pi/4$ apart, they can measure, and no matter which outcome they get, the sign in the inner product will always by $1/\sqrt{2}$.

4: An easy way to see this is to note that $P_\pm = \frac{I \pm \sigma_x}{2}$. Using this, we can show the equivalence of the two representations:

$$\begin{aligned}(1-2p)\rho + 2pP_-\rho P_- + 2pP_+\rho P_+ &= (1-2p)\rho + \frac{p}{2}(I-\sigma_x)\rho(I-\sigma_x) + \frac{p}{2}(I+\sigma_x)\rho(I+\sigma_x) \\ &= (1-p)\rho + p\sigma_x\rho\sigma_x. \end{aligned} \tag{25}$$

5: Note that the eigenvectors of $X_1X_2X_3$ are $\frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$ with eigenvalues ± 1. From this we can see that the eigenvectors of $X_1X_2X_3X_4X_5X_6$ with eigenvalue $+1$ are $\frac{1}{2}(|000\rangle \pm |111\rangle)(|000\rangle \pm |111\rangle)$ and the eigenvectors with eigenvalue -1 are $\frac{1}{2}(|000\rangle \pm |111\rangle)(|000\rangle \mp |111\rangle)$. Thus, we measure a $+1$ when the phases of the two blocks are the same, and a -1 when they are opposite. By the same argument $X_4X_5X_6X_7X_8X_9$ compare the phases of the second and third blocks. Thus, the four outcomes $\{1, -1\} \otimes \{1, -1\}$ from measuring these two observables correspond to phase flips on any of the 9 qubits.

6: First, note that the error is the same no matter which of the first three qubits receives the phase flip: $Z_i(|000\rangle \pm |111\rangle) = (|000\rangle \mp |111\rangle)$ for $i = \{1, 2, 3\}$. From this, we see that the recovery operation $Z_1Z_2Z_3$ flips the phase three times, which is equivalent to flipping it once. This illustrates why the Shor code is a *degenerate* code, as is discussed in the textbook.

7a: The outcome of a single rotation is $\cos\theta\,|0\rangle + \sin\theta\,|1\rangle$. If we rotate n times, the state is $\cos n\theta\,|0\rangle + \sin n\theta\,|1\rangle$. Thus for the probability of measuring a $|1\rangle$ to be approximately $\frac{1}{4}$, we need $\sin^2 n\theta \approx \frac{1}{4} \Rightarrow n\theta \approx \frac{\pi}{6} \Rightarrow n \approx \frac{\pi}{6\theta}$.

7b: If we measure after each rotation, we are in fact performing a bit flip operation with the probability of flipping $p = \sin^2\theta$. We are looking for the number of trials before the probability of a single flip is approximately $\frac{1}{4}$. This is easiest done if we look for the number of trials where the probability of not flipping is $\frac{3}{4}$.

$$\frac{3}{4} \approx \cos^{2n}\theta \tag{26}$$

$$\Rightarrow \log\frac{3}{4} \approx 2n\log(\cos\theta) \tag{27}$$

$$\Rightarrow n \approx \frac{\log\frac{3}{4}}{2\log(\cos\theta)} \tag{28}$$

$$\approx \frac{.27}{\theta^2} \tag{29}$$

The last line comes from using the small angle approximation for $\cos\theta$ and the Taylor expansion for $\log(1 - \frac{\theta^2}{2}) \approx -\frac{\theta^2}{2}$.