18.435/2.111 Homework # 10

Due Thursday, December 7

1: Suppose that in a quantum error-correcting code which is able to correct one error, the second qubit is put though an amplitude damping channel, and all other qubits are left untouched. That is, the second qubit undergoes the quantum operation

$$\rho \rightarrow E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger,$$

where

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

The decoding procedure then measures which error has occurred. With some probabilities, this measurement will yield either no error, or a σ_x, σ_y or σ_z error on the second qubit. What are these four probabilities?

The next few exercises are related to the Peres-Wootters paper I talked about the Tuesday before Thanksgiving. Let

$$|\, v_\theta \rangle = \cos\theta \,|\, 0 \rangle + \sin\theta \,|\, 1 \rangle\,.$$

Note that $|\, v_\theta \rangle = -\,|\, v_{\theta+\pi} \rangle$. and $\langle v_{\theta+\pi/2} | v_\theta \rangle = 0$.

2a: Suppose that we have the state on two qubits $|\, v_\theta \rangle \otimes |\, v_\theta \rangle$. Show that this state is orthogonal to the state $\frac{1}{\sqrt{2}}(|\, 01 \rangle - |\, 10 \rangle)$.
2b: Compute the density matrices

$$\frac{1}{\pi} \int_{\theta=0}^{\pi} |\, v_\theta \rangle \,|\, v_\theta \rangle \,\langle v_\theta \,|\, \langle v_\theta \,|\, d\theta.$$

and

$$\frac{1}{\pi} \int_{\theta=0}^{\pi} |\, v_\theta \rangle \,|\, v_\theta \rangle \left\langle v_{\theta+\pi/2} \right| \left\langle v_{\theta+\pi/2} \right| d\theta.$$

Express the results as a sum of states $|\, \beta_i \rangle\langle \beta_i \,|$ where $|\, \beta_i \rangle$ are the four Bell states. (To check your results, note that the trace of the answers should be 1 and 0, respectively.)
2c: For a continuous POVM, we replace the sum in the formulation you are used to by an integral, which must be equal to the identity. Show that if we let

$$|\, e_\theta \rangle = s \,|\, v_\theta \rangle \,|\, v_\theta \rangle + t \left| v_{\theta+\pi/2} \right\rangle \left| v_{\theta+\pi/2} \right\rangle$$

then we can choose s and t so that

$$\frac{1}{\pi} \int_{\theta=0}^{\pi} |\, e_\theta \rangle\langle e_\theta \,|\, d\theta = I - \frac{1}{2}(|\, 01 \rangle - |\, 10 \rangle)(\langle 01 \,| - \langle 10 \,|).$$

(Problem 2b may be of help, although there are other, quite different, ways of showing this).

The matrices $\frac{1}{3}\,|\,e_\theta\rangle\langle e_\theta\,|$ for $\theta = 0,\ \pi/3,\ 2\pi/3$, are the three projectors in the optimal joint measurement discussed in the Peres and Wootters paper.

There is actually a POVM all of whose elements are tensor products which is equivalent on states $|\,v\rangle \otimes |\,v\rangle$ to the measurement given by 2c. In problems 3a-3c we will show this.

3a: Show that if we choose α properly and take $|\,e_0\rangle + \frac{\alpha}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$, we obtain a tensor product state $c\,|\,v_{\theta_1}\rangle \otimes |\,v_{\theta_2}\rangle$. Here $|\,e_0\rangle$ is the state from 2c with $\theta = 0$ and s, t set to the values you calculated in 2c. What are c, α, and $\langle v_{\theta_1}|v_{\theta_2}\rangle$?

Of course, the above calculation works for arbitrary θ, and not just $\theta = 0$. You can use this to show (assuming $|\alpha| \leq 1$) that the POVM outcomes above are equivalent to a POVM all of elements are tensor products, i.e., $|\,v_{\theta_1}\rangle\langle v_{\theta_1}\,| \otimes |\,v_{\theta_2}\rangle\langle v_{\theta_2}\,|$. This means that in the continuous version of the Peres-Wootters situation, where Alice and Bob are given two identical quantum systems each in state $|\,v_\theta\rangle$ with θ is chosen uniformly betweeen 0 and π, Alice and Bob can use local quantum operations and classical communication to achieve an optimal measurement. It is only when they are given states not chosen uniformly from all θ that they need to make a joint measurement on their states to achieve the best possible outcomes.

3b: Why did I require $|\alpha| \leq 1$ above?

3c: How can Alice and Bob use only classical communication to make measurements on a pair of qubits so that they always obtain two POVM operators $|\,v_{\theta_1}\rangle\langle v_{\theta_1}\,|$ and $|\,v_{\theta_2}\rangle\langle v_{\theta_2}\,|$ so

$$|\langle v_{\theta_1}|v_{\theta_2}\rangle|$$

is the optimal value calculated in 3a.

4 - 6: Do Exercises 10.2, 10.5 and 10.6 in Nielsen and Chuang.

7: Suppose we have the unitary transformation

$$U := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{pmatrix}$$

with θ small. We apply U to a qubit which starts off in state $|\,0\rangle$.

7a Approximately how many times do we need to apply U before a measurement of the qubit would have a reasonable (say $\frac{1}{4}$) chance of observing a $|\,1\rangle$?

7b Suppose we measure the qubit in the $|\,0\rangle$, $|\,1\rangle$ basis after each application of U. Approximately how many times do we need to apply U before the qubit has a reasonable (say $\frac{1}{4}$) chance of being a $|\,1\rangle$?

The phenomenon in problem 7 is called the *quantum Zeno effect* and occurs if U is close to the identity, and also in continuous evolution of quantum systems.