18.435/2.111 Homework # 5

Due Tuesday, November 25

1. What is the density matrix obtained if you have a qubit which is in state

$$\begin{array}{rll} |\,0\rangle & \text{with probability} & \frac{1}{3}, \\ -\frac{1}{2}\,|\,0\rangle + \frac{\sqrt{3}}{2}\,|\,1\rangle & \text{with probability} & \frac{1}{3}, \\ -\frac{1}{2}\,|\,0\rangle - \frac{\sqrt{3}}{2}\,|\,1\rangle & \text{with probability} & \frac{1}{3}. \end{array}$$

2. What is the density matrix obtained if you take the partial trace over the second qubit of the following state; i.e., what is

$$\mathrm{Tr}_2\,\frac{1}{\sqrt{3}}(|\,00\rangle + |\,01\rangle + |\,10\rangle).$$

3. One way to obtain a noisy quantum operation is to have the input quantum state interact with another "environment" quantum system, and then take a partial trace that removes the "environment" system.

Suppose we start with a qubit in state $|\,\psi\rangle$, and an "environment qubit" $|\,e\rangle$ in state $\frac{\sqrt{3}}{2}\,|\,0\rangle + \frac{1}{2}\,|\,1\rangle$. We then apply the quantum gate controlled σ_z

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

to the state $|\,\psi\rangle \otimes |\,e\rangle$, and take the partial trace to remove $|\,e\rangle$. Express the resulting quantum operation in operator sum notation,

$$\rho \to \sum_i A_i \rho A_i^\dagger.$$

4. Suppose we start with a qubit and first apply the dephasing operation

$$\rho \to (1-p)\rho + p\sigma_z\rho\sigma_z^\dagger$$

and then apply the amplitude damping operation

$$\rho \to \sum_{i=1}^{2} A_i \rho A_i^\dagger$$

where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-q} \end{pmatrix} \qquad\qquad A_2 = \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix}.$$

Show the resulting transformation can be expressed in the operator-sum notation with just three matrices A_i:

$$\rho \rightarrow \sum_{i=1}^{3} A_i \rho A_i^\dagger.$$

5. Consider the depolarizing quantum operation $\mathcal{D}$:

$$\mathcal{D}(\rho) = (1-p)\rho + \frac{p}{3} \sum_{a=x,y,z} \sigma_a \rho \sigma_a^\dagger,$$

with $p < 3/4$. Suppose we apply $\mathcal{D}$ to a density matrix ρ_{in} to obtain $\rho_{\text{out}} = \mathcal{D}(\rho_{\text{in}})$. Show that the minimum possible eigenvalue of a density matrix output from this operation is $2p/3$.
Hint: use the identity

$$\frac{1}{4}\rho + \frac{1}{4} \sum_{a=x,y,z} \sigma_a \rho \sigma_a^\dagger = \frac{I}{2},$$

where I is the identity matrix.

6. Do Exercise 10.27 in Nielsen and Chuang.
Hint: one way to do this is to show directly that it has equivalent error-correcting properties for bit errors, and then take the Hadamard transform of the code and show that this works well for phase errors. Another approach is to follow the proof that Nielsen and Chuang use to derive the phase error-correcting properties of a CSS code.