18.435/2.111 Homework # 4

Due Thursday, November 6

1. In the teleportation protocol, show that the probability distribution for the values of the two qubits that Alice sends to Bob is independent of the state $|\,\psi\rangle$ of the qubit being transmitted. In other words, an eavesdropper can infer nothing about the value of $|\,\psi\rangle$ by knowing the values of the two classical bits transmitted.

2. Show that Alice can teleport two qubits $|\,\phi\rangle\,|\,\psi\rangle$ "through" the gate

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

so that Bob obtains two qubits in the state $S\,|\,\phi\rangle\,|\,\psi\rangle$.

More specifically, suppose Alice and Bob share four qubits in the state

$$\frac{1}{2}\Big(\,|\,0000\rangle + |\,0011\rangle + |\,1100\rangle - |\,1111\rangle\,\Big),$$

with Alice holding the first pair of qubits and Bob holding the second pair. Alice measures $|\,\phi\rangle$ and her first qubit in the Bell basis, and measures $|\,\psi\rangle$ and her second qubit in the Bell basis. Alice sends the results of these measurements to Bob over a classical channel, and Bob applies some unitary transformations, which depend on the classical information he receives from Alice, to his two qubits to obtain the state $S\,|\,\phi\rangle\,|\,\psi\rangle$. Explain how the transformations Bob applies depend on the results of Alice's measurements.

3. Generalize the superdense coding procedure to three-dimensional quantum states (qutrits). Let $|\,0\rangle\,,|\,1\rangle\,,|\,2\rangle$ be an orthonormal basis for the qutrits. Now, suppose Alice and Bob share a pair of qutrits in the state

$$|\,\mathrm{EPR}_3\rangle = \frac{1}{\sqrt{3}}\Big(\,|\,00\rangle + |\,11\rangle + |\,22\rangle\,\Big)$$

Show that there are 9 unitary operations so that if Alice performs one of these unitary operations on her half of the state $|\,\mathrm{EPR}_3\rangle$, and sends the resulting qubit to Bob, he can then make a measurement on the two qutrits that he now holds which deterministically tells him which operation Alice performed. This shows that superdense coding can encode $\log_2 9$ bits in one qutrit.
Hint: Define the two matrices

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

Powers of these of the form R^aT^b with $0 \le a, b \le 2$ will play the role of the Pauli matrices in this superdense coding procedure, with the Bell basis replaced by the states $R^aT^b\,|\,\mathrm{EPR}_3\rangle$. You will need to show that these states form an orthonormal basis of the space of two qutrits.

4. Suppose that we have four parties, Alice, Bob, Cathy, and David. Alice and Cathy share a pair of qubits which are in one of the four Bell basis states,

$$\frac{1}{\sqrt{2}}(|\,01\rangle \pm |\,10\rangle), \qquad\qquad \frac{1}{\sqrt{2}}(|\,00\rangle \pm |\,11\rangle),$$

but they don't know which state it's in. Bob and David share a pair of qubits in the same Bell state. Suppose further that Alice and Bob share a pair of qubits in the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$. Show that there is a protocol that lets Cathy and David end up sharing a pair of qubits in the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$.
Hint: Let Alice teleport her half of the Alice-Bob pair to Cathy, and Bob teleport his half of this pair to David. Show that even though Cathy and David don't know which Bell state they had, they can still apply Pauli transformations to their teleported qubits to end up with the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$.

5. (Extra Credit) Show that in the model for quantum computation using measurement on cluster states, a SWAP gate can be implemented by bringing two transmission lines together for three qubits, and measuring every qubit using the observable σ_x (i.e., in the basis $\frac{1}{2}(|\,0\rangle \pm |\,1\rangle)$). For example, the following cluster

$$\begin{array}{ccccccc}
\sigma_x & \sigma_x & \sigma_x & \sigma_x & \sigma_x & \sigma_x & \sigma_x \\
 & & \sigma_x & \sigma_x & \sigma_x & & \\
\sigma_x & \sigma_x & \sigma_x & & \sigma_x & \sigma_x & \sigma_x
\end{array}$$

would implement a SWAP gate.