18.435/2.111 Homework # 3

Due Thursday, October 16

The first three problems relate to the factoring algorithm. Recall that we factored N by constructing the unitary transformation U which takes $U\,|\,a\rangle = |\,ax \bmod N\rangle$ for $0 \le a < N$ and $\gcd(x, N) = 1$. We found the minimum $r > 0$ for which $U^r\,|\,1\rangle = |\,1\rangle$ and used it to factor N. Note that for some of these problems, Theorem A4.10 on page 632 of N&C may come in handy. This theorem says that the multiplicative group of residues mod p^α is cyclic for odd primes p.

1. For $N = 15$, what fraction of the residues $1 \le x < N$ with $\gcd(x, N) = 1$ will result in a factorization? How about for $N = 63$? (While testing all these residues is one way to solve this problem, there are much more efficient ones.)

2. Suppose we try to apply the factoring algorithm to a number $N = p^\alpha$ which is a power of p. Will it work? If not, what goes wrong?

3. Suppose we try to apply the factoring algorithm, but we forget to check whether $\gcd(x, N) = 1$ and accidentally choose an x with $1 < x < N$ and $\gcd(x, N) > 1$. Will the algorithm still work? If not, what goes wrong?

The next two problems deal with the period-finding algorithm on p. 236 of N&C. This was not covered in class, but is quite similar to the order-finding algorithm (p. 232) which was. The difference is that the order-finding algorithm operates on a black box U which performs $U\,|\,a\rangle = |\,f(a)\rangle$ where f is a classical one-to-one function, and finds the minimum value of r such that $U^r\,|\,b\rangle = |\,b\rangle$, whereas the period-finding algorithm operates on a black box U such that $U\,|\,x\rangle\,|\,y\rangle = |\,x\rangle\,|\,y \oplus f(x)\rangle$. Also note that the value of t is given using big-O notation, but for the algorithm to work, you actually need $t \ge 2L$.

4. Do Exercise 5.20 in N&C.

5. Suppose we apply this period-finding algorithm to the function

$$\begin{aligned} f(x) &= 1 \qquad \text{if } r \text{ divides } x \\ f(x) &= 0 \qquad \text{if } x \text{ is not a multiple of } r. \end{aligned}$$

Approximately what is the probability that we learn the period r?

6. For Grover's search algorithm, assume that we have M target states out of N total states, so the black box O takes

$$\begin{aligned} O\,|\,x\rangle &= -\,|\,x\rangle \qquad \text{if } x \text{ is a target state,} \\ O\,|\,x\rangle &= |\,x\rangle \qquad \text{otherwise.} \end{aligned}$$

Suppose we find a target state with probability 1 after one iteration of the algorithm. What can you say about the ratio M/N?

7. Consider the modification to Grover's algorithm so that the oracle now performs

$$\begin{aligned} O\,|\,x\rangle &= e^{i\phi}\,|\,x\rangle && \text{if } x \text{ is a target state,} \\ O\,|\,x\rangle &= |\,x\rangle && \text{otherwise.} \end{aligned}$$

Show that if you use the transformation

$$\tilde{G} = H^{\otimes n}\left[(1 - e^{i\phi})\,|\,0\rangle\,\langle 0\,| - I\right] H^{\otimes n} O$$

instead of the standard Grover iteration, for any state with M/N sufficiently large you can choose ϕ so that the algorithm finds a target state with probability 1 after one iteration. For what values of M/N is there such a ϕ?