Polytope Sparsification

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Joint work with

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(Penn)

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(CMU)

Rocco Servedio
(Columbia)

Motivation from the Boolean Setting

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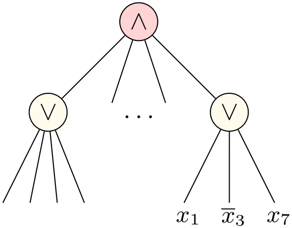

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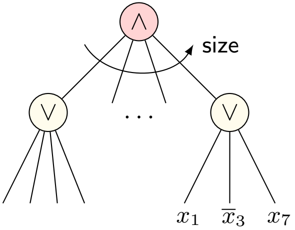

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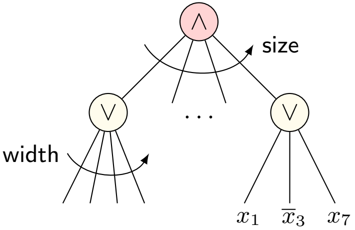

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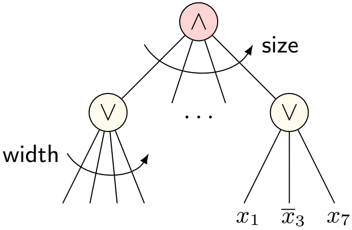

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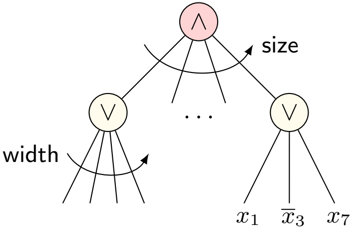

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The Hamming Distance

Definition: Given $f, g : \{0,1\}^n \to \{0,1\}$, we define

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We will write $f \approx_\varepsilon g$ if $\mathrm{dist}(f,g) \leq \varepsilon$.

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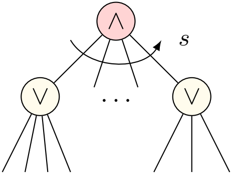

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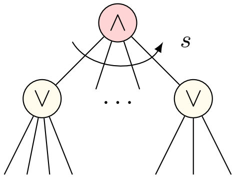

$\approx_\varepsilon$

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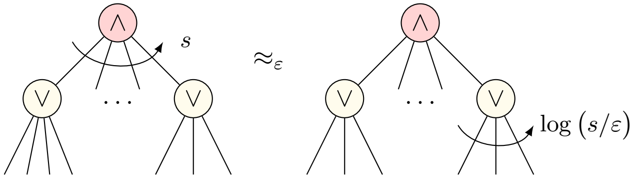

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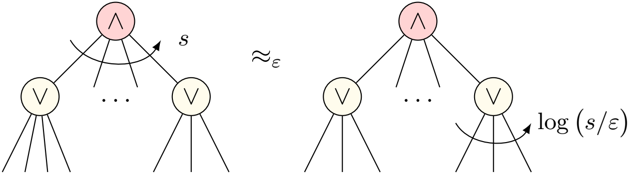

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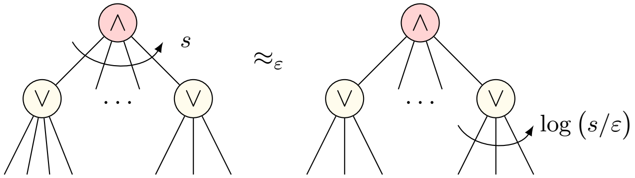

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Idea: Drop wide clauses, take union bound ☺

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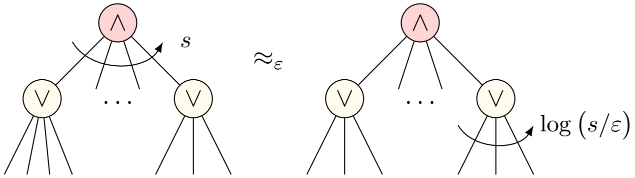

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What about a converse?

Are Narrow CNFs (Essentially) Small?

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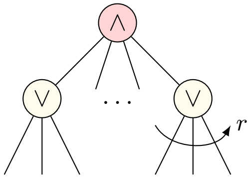

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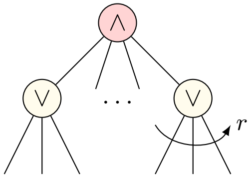

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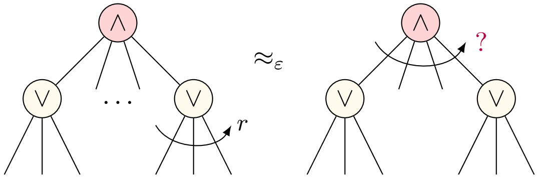

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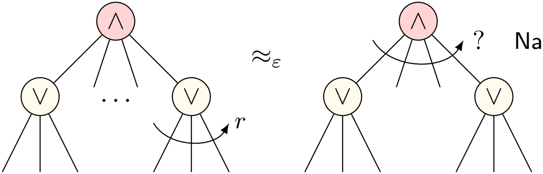

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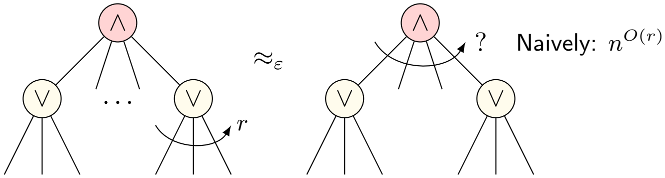

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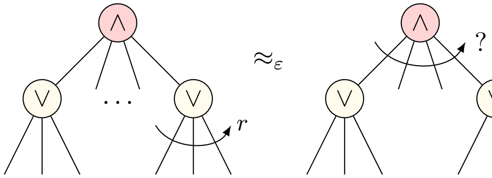

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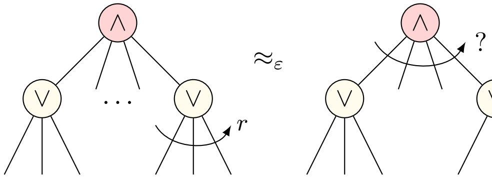

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Hope: Is every width-r CNF $\approx_\varepsilon$ size-$O_{\varepsilon,r}(1)$ CNF?

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$\left(2 + \frac{1}{r} \log\left(\frac{1}{\varepsilon}\right)\right)^{O(r)}$ — [Lovett-Wu-Zhang '21]
"Mild" random restrictions + noise stability

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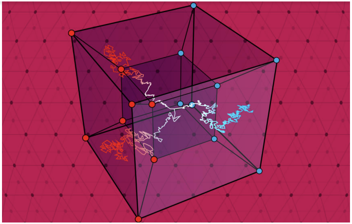
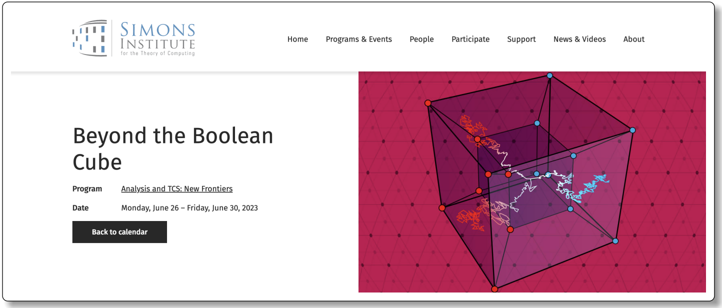
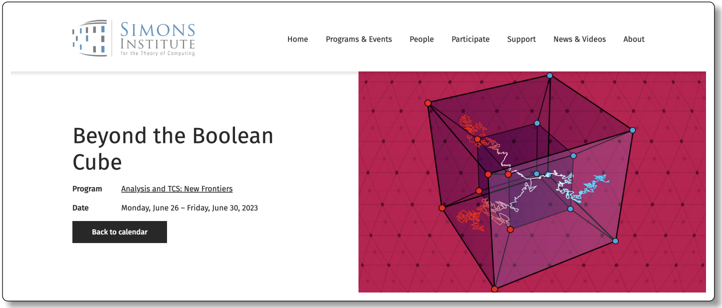




Analogue of CNF sparsification over Gaussian space?



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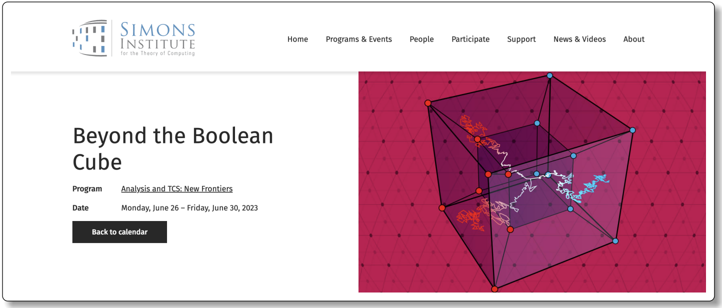

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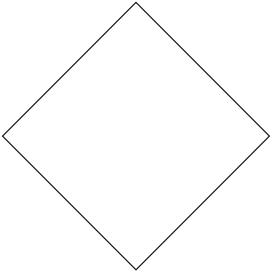
Narrow Clauses



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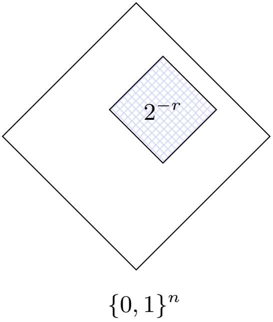
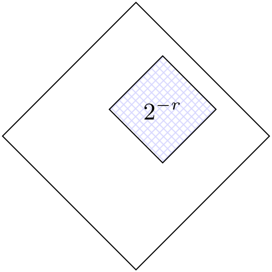


Clause of width r

e.g. $x_1 \vee \overline{x}_2 \longleftrightarrow 01 * \cdots *$

Narrow Clauses ≈ Narrow Halfspaces



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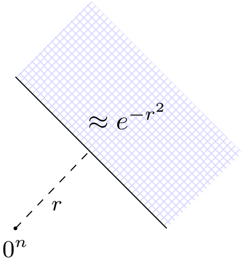

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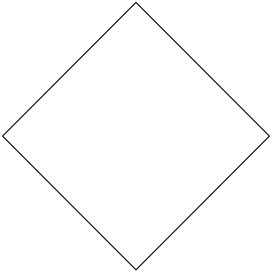
Halfspace of width r

e.g. $\mathbf{1}\{x \cdot v \leq 10\}$ for $v \in \mathbb{S}^{n-1}$

Narrow CNFs



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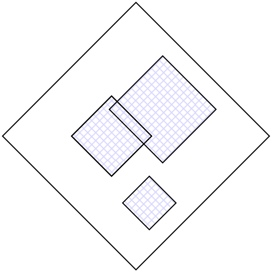
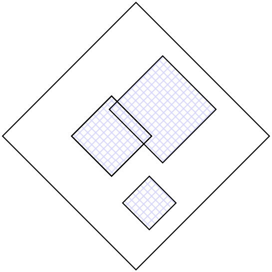


$\{0,1\}^n$

AND of width-r ORs

Narrow CNFs ≈ Bounded-Width Polytopes



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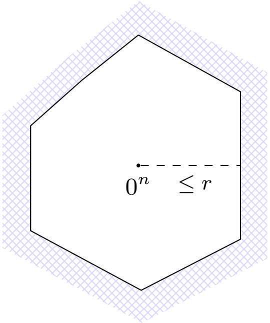
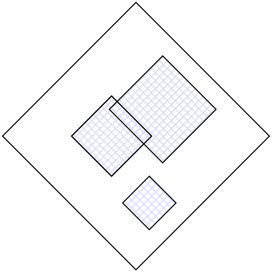


AND of width-r ORs

Intersection of width-r halfspaces

Narrow CNFs $\approx$ Bounded-Width Polytopes



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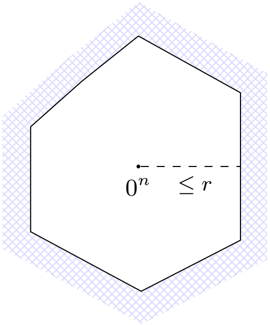

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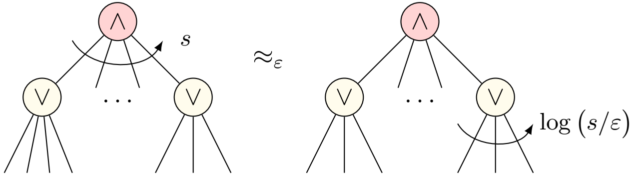
Width-r polytope

Recall: Small CNFs are (Essentially) Narrow



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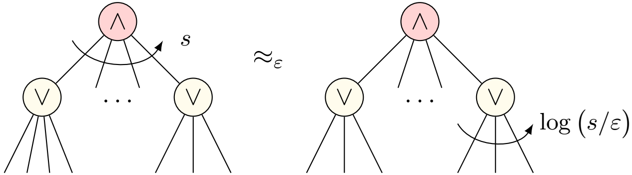

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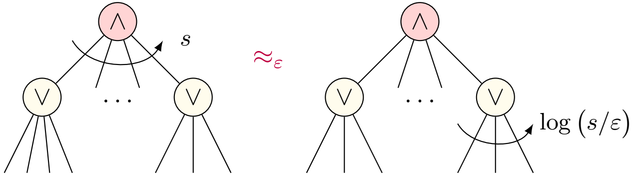

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Note: Not translation invariant!

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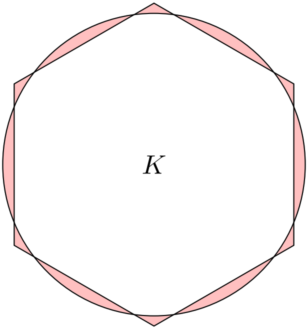

$L = H_1 \cap \ldots \cap H_6$

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Intersections of Few Halfspaces are (Essentially) Narrow

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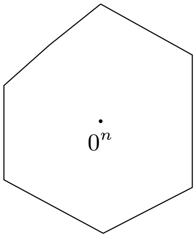

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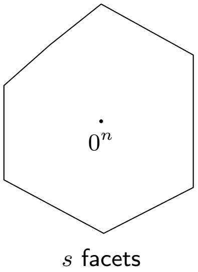

$\approx_\varepsilon$

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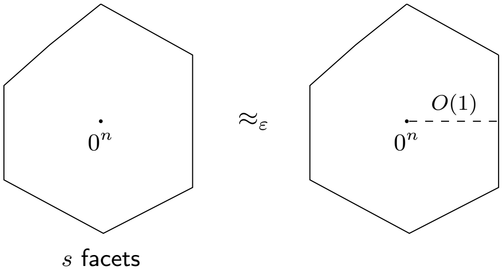

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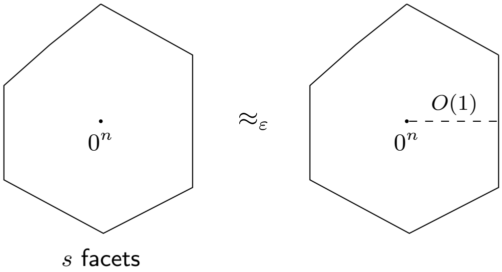

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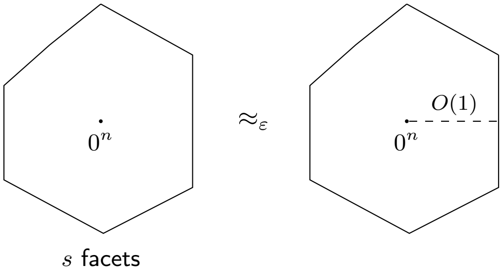

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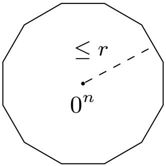

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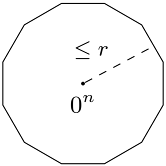

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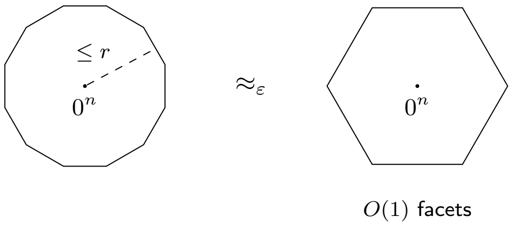

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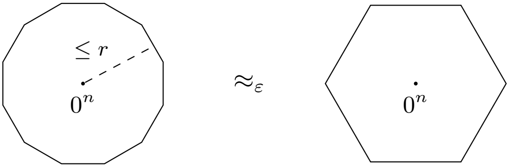

$O(1)$ facets

[DNOS'25]: If $K \subseteq \mathbb{R}^n$ is a polytope of width r, then $K \approx_\varepsilon L$ where L is an intersection of $\exp\exp(r^4/\varepsilon^2)$ halfspaces.

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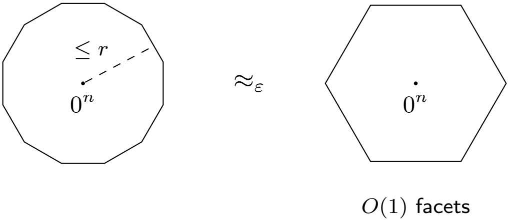

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Yields efficient learning & tolerant testing algorithms

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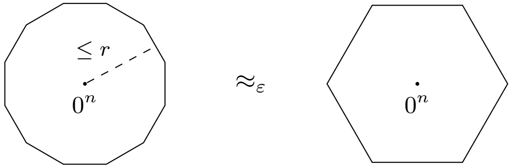

$O(1)$ facets

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CNF Sparsification: A Brief History

Width-r CNFs are ε-approximated by CNFs of size:

$2^{O\left(r2^r \log\left(\frac{1}{\varepsilon}\right)\right)}$ [Trevisan '05]
Covering/packing argument

$\left(r \log\left(\frac{1}{\varepsilon}\right)\right)^{O(r)}$ [Gopalan-Meka-Reingold '13]
Quasi-sunflowers

$\left(\frac{1}{\varepsilon}\right)^{O(r)}$ [Lovett-Zhang '18]
Switching lemma + noise stability

$\left(2 + \frac{1}{r} \log\left(\frac{1}{\varepsilon}\right)\right)^{O(r)}$ [Lovett-Wu-Zhang '21]
"Mild" random restrictions + noise stability

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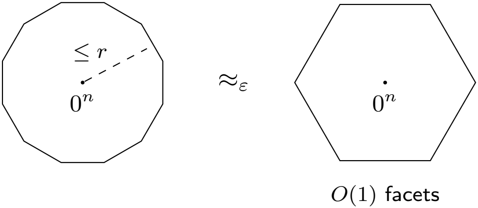

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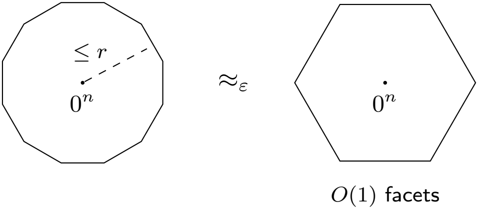

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Lower Bound: $\exp(1/\varepsilon)$ halfspaces necessary

From Polytopes to Suprema

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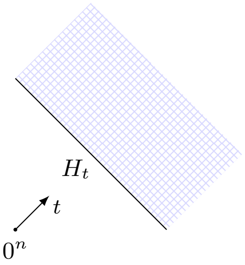

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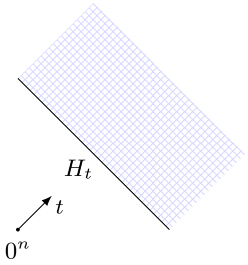

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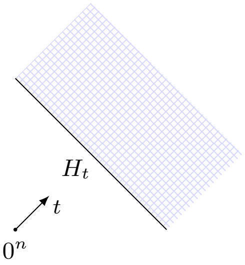

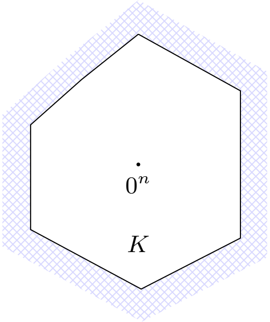

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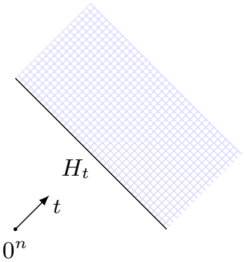

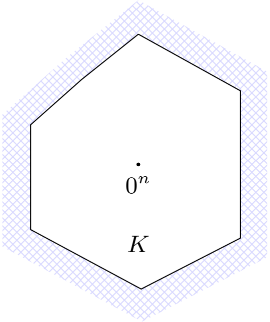

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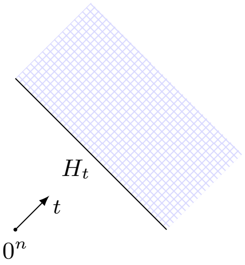

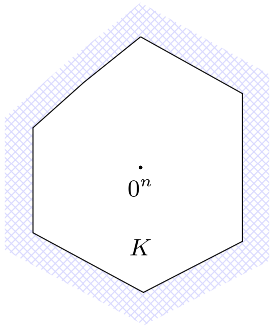

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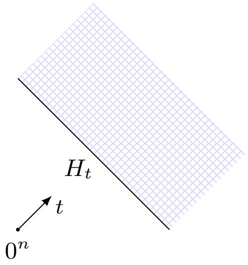

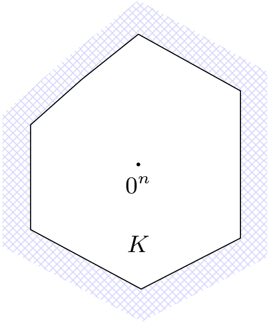

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Compressed sensing, metric embeddings, random matrix theory, ...

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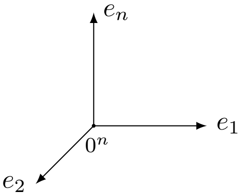

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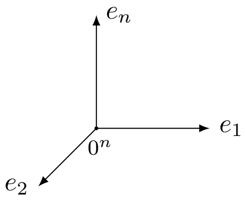

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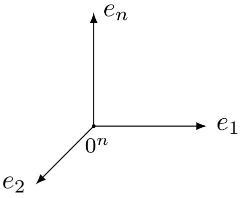

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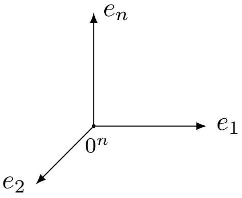

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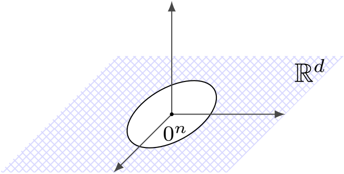

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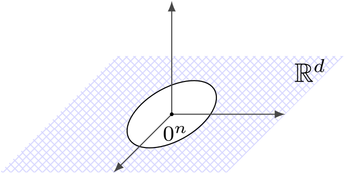

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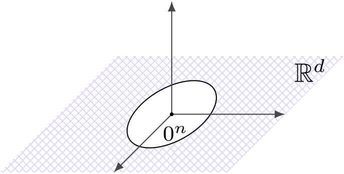

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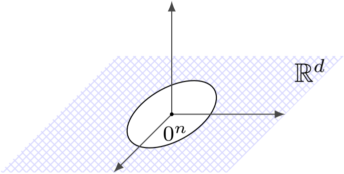

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Informally: Complexity of T determines $\mathbf{E}\left[\sup_{t \in T} \boldsymbol{X}_t\right]$

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Note: Shifts c_s are necessary 😕

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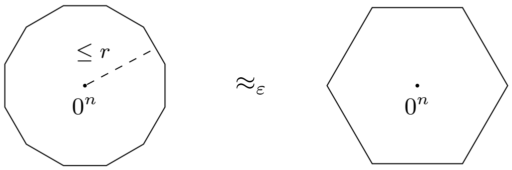

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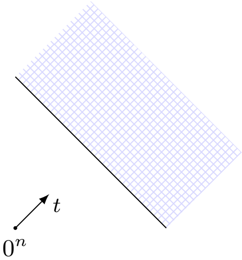

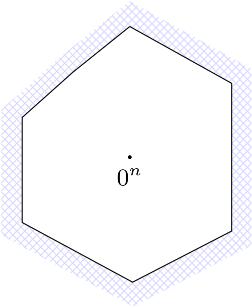

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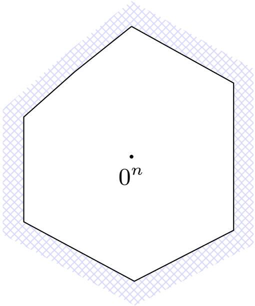

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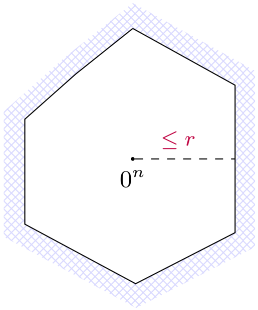

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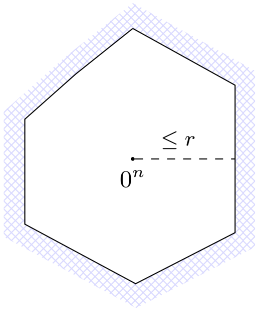

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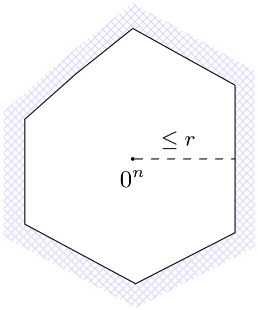

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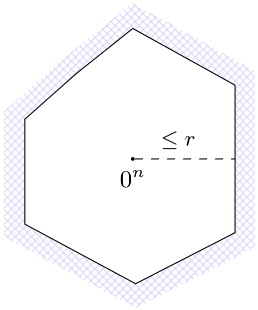

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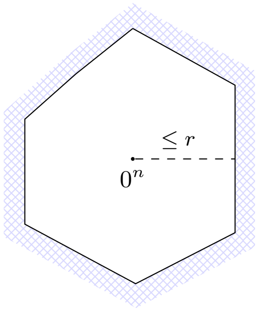

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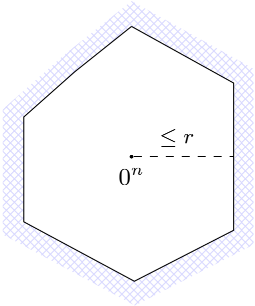

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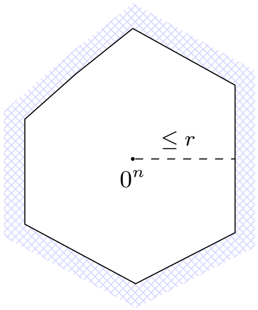

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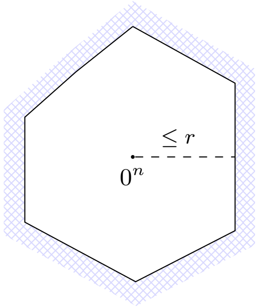

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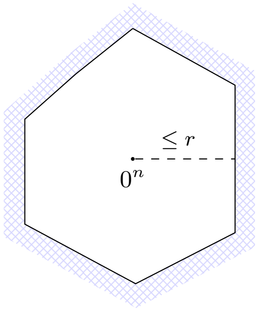

$$K = \bigcap_{t\in T} H_t$$

$$x \in K \iff \sup_{t\in T} \langle x, t\rangle \le 1$$

Anti-concentration: $K \approx_\varepsilon L$

Back to Our Main Result

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[DNOS'25]: Let $T \subseteq \mathbb{R}^n$ and let $\{\boldsymbol{X}_t\}_{t \in T}$ be the associated Gaussian process. For any $\varepsilon \in (0, 0.5)$, there is a subset $S \subseteq T$ and constants $\{c_s \geq 0\}_{s \in S}$ such that

$$\mathbf{E}\left[\left|\sup_{t \in T} \boldsymbol{X}_t - \sup_{s \in S}\{\boldsymbol{X}_s + c_s\}\right|\right] \leq \varepsilon\, \mathbf{E}\left[\sup_{t \in T} \boldsymbol{X}_t\right],$$

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Idea: $\phi(x) := \sup_{t \in T} \langle x, t \rangle$ for symmetric T

Recap

The Main Technical Result

Sparsifying Suprema of Gaussian Processes

Some Consequences

Polytope Sparsification, Norms are (Essentially) Juntas

Recap & Rest of Talk

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Elements of the Proof

Talagrand's Majorizing Measures Theorem

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First: Rescale so that $\mathbf{E}[\sup_{t \in T} \boldsymbol{X}_t] = 1$

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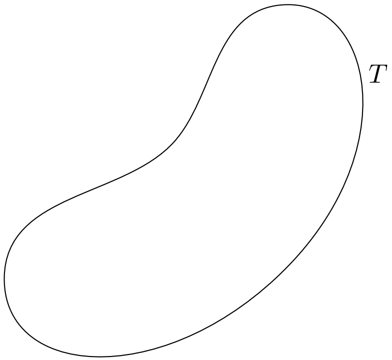
Natural Idea: Cluster the points in T

Attempt 1: A δ-Cover



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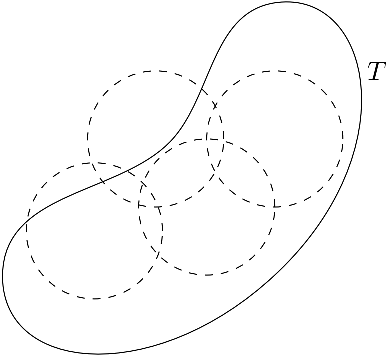

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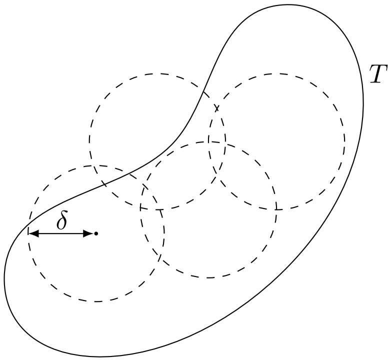

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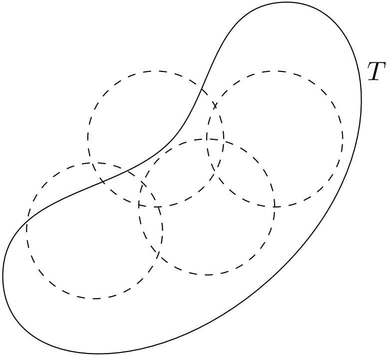

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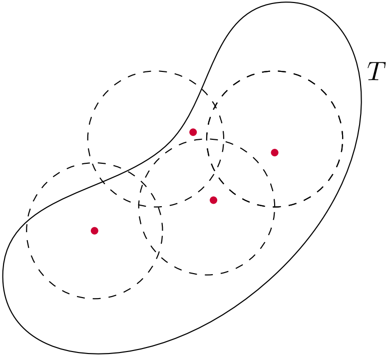

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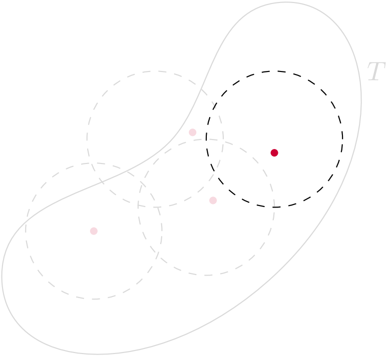

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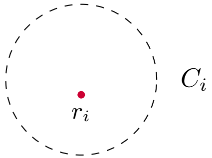

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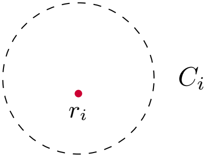

$$\sup_{t \in C_i} \langle \boldsymbol{x}, t \rangle$$

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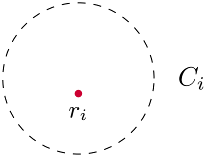

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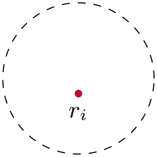

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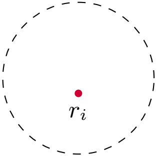

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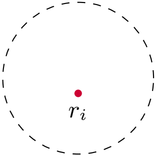

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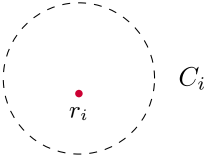

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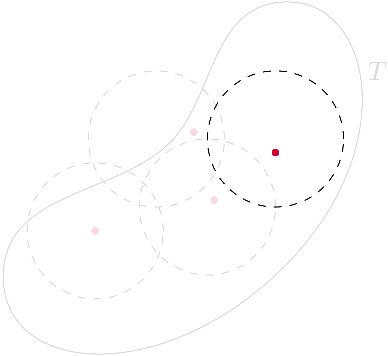
(Consequence of Lipschitz concentration of Gaussians)

Approximation Across M Clusters



Within each cluster C_i:

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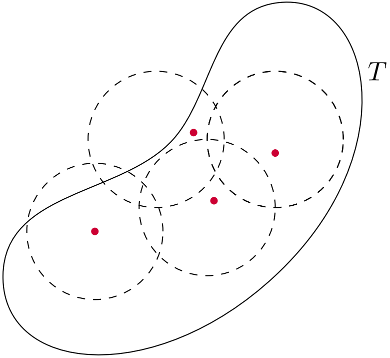

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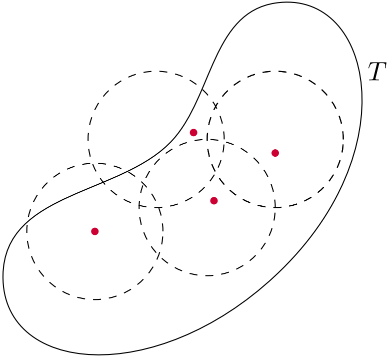

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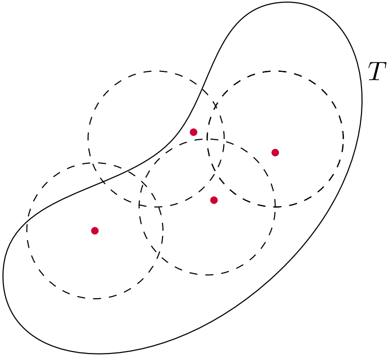

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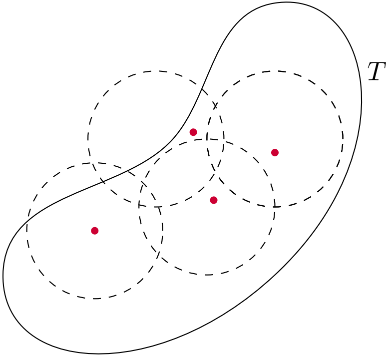

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Size $= M$ Error $\asymp \delta \sqrt{\log M}$

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Set δ as an appropriate polynomial of ε ☺

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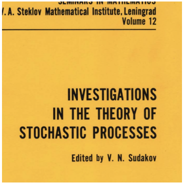
Sudakov Minoration



Theorem: If smallest δ net for $T \subseteq \mathbb{R}^n$ has size M, then

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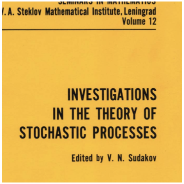

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Recall: We rescaled to ensure LHS $= 1$

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Issue: Gaussian processes s.t. the best δ-covering of the set has size $2^{\Theta(1/\delta^2)}$ at every fixed scale δ ☹


20
E3-20S
ESTWING

Talagrand's Majorizing Measures Theorem

Gives hierarchical sequence of partitions of T:

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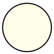

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Talagrand's Majorizing Measures Theorem

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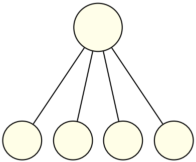

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Refinement $\downarrow$

$\mathcal{A}_1$

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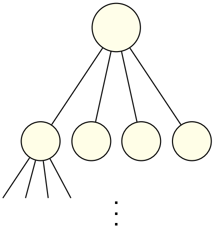

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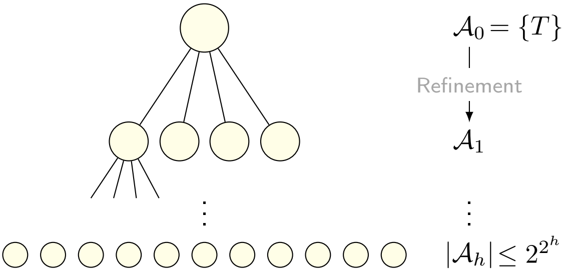

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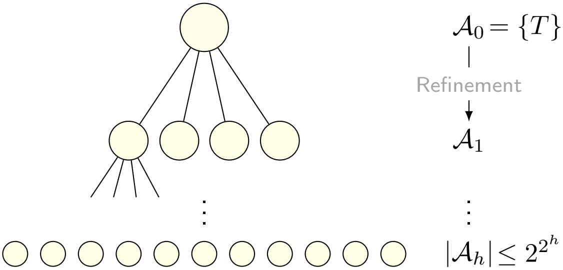

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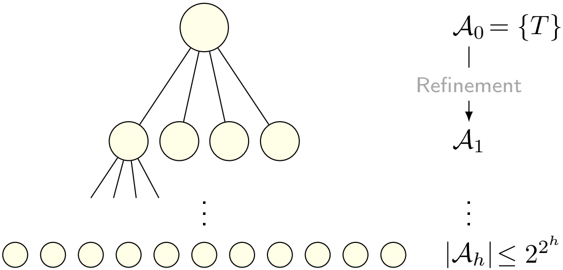

$$\mathbf{E}\left[\sup_{t \in T} \boldsymbol{X}_t\right] \geq \sum_{h \geq 0} 2^{h/2} \cdot \mathrm{diam}(\mathcal{A}_h(t)) \qquad \text{for all } t \in T$$

Our Proof from 30,000 Feet

Recall we have Gaussian process $\{\boldsymbol{X}_t\}_{t \in T}$ with $\mathbf{E}[\sup \boldsymbol{X}_t] = 1$

Thanks to rescaling assumption

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See paper for full details (and more) ☺

Recap

The Main Technical Result

Sparsifying Suprema of Gaussian Processes

Some Consequences

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Thanks for listening! Questions? ☺