Approximating Sumset Size

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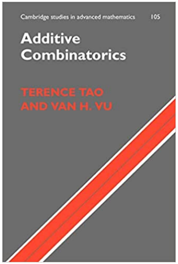
Sumsets

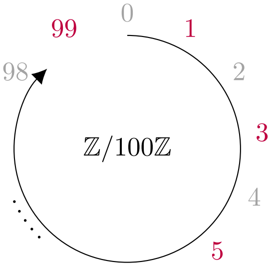
Definition: Given an abelian group $(G, +)$ and a subset $A \subseteq G$, we define the sumset $A + A$ as

$$A + A := \{a + b : a, b \in A\}.$$

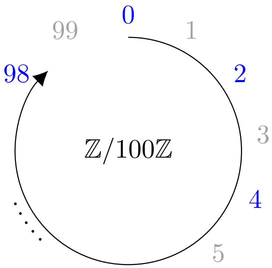
- Note $A + A \neq 2A := \{a + a : a, a \in A\}$.
- Fundamental object of study in additive combinatorics.



Easy Example



$A = \{1, 3, \ldots, 99\}$



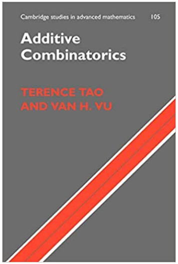
$A + A = \{0, 2, \ldots, 98\}$

- Note that $|A| = |A + A|$.
- A is a coset of the subgroup of even residues modulo 100.

Why Sumset Size?

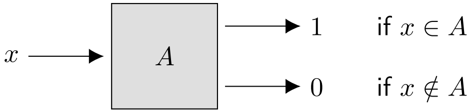
Easy Exercise: For $A \subseteq G$, if $|A| = |A + A|$, then $A = x + H$ for some subgroup $H \leq G$ and $x \in G$.

Robustifications & Variants



Freiman–Rusza, Plünneke–Rusza, Balog–Szemerédi–Gowers, etc.

A Natural Question



Question: Given query access to $A \subseteq G$, what is $\frac{|A+A|}{|G|}$ up to an error of $\pm\varepsilon$?

$\mathrm{Vol}(A+A)$

This work: $\mathbb{F}_2^n$

A Natural Question over $\mathbb{F}_2^n$

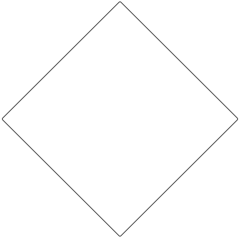
Question: Given query access to $A \subseteq \mathbb{F}_2^n$ and writing

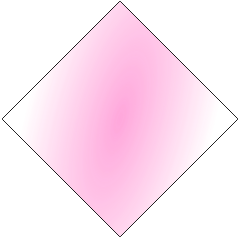
$$\mathrm{Vol}(A) := \frac{|A|}{2^n},$$

what is $\mathrm{Vol}(A+A)$ up to an error of $\pm\varepsilon$?

- Cost measure: number of queries (as a function of n and ε).
- At first glance: To confirm $z \notin A+A$, have to check that at least one of $x, y \notin A$ for the 2^n pairs (x, y) satisfying $x + y = z$.

No Query-Efficient Algorithm over $\mathbb{F}_2^n$





$A = \emptyset$

$\boldsymbol{A}$ is a random set of size $2^{0.51n}$

$\mathrm{Vol}(A + A) = 0$

$\mathrm{Vol}(\boldsymbol{A} + \boldsymbol{A}) \geq 1 - \exp(-n)$ w.h.p.

Need $\Omega(2^{0.49n})$ queries to distinguish A from $\boldsymbol{A}$.

Refining The Original Question

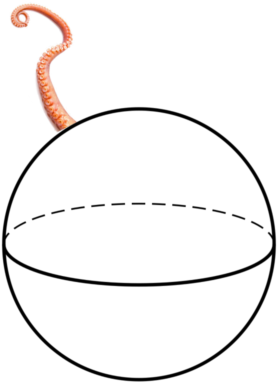
Original Question: Given query access to $A \subseteq \mathbb{F}_2^n$ and writing

$$\mathrm{Vol}(A) := \frac{|A|}{2^n},$$

what is $\mathrm{Vol}(A+A)$ up to an error of $\pm\varepsilon$?

- Adding a small (random) collection $R \subseteq \mathbb{F}_2^n$ of $2^{0.51n}$ elements to A can blow up $\mathrm{Vol}(A+A)$ to almost 1.
- Natural relaxation: Output $\mathrm{Vol}(A'+A')$ for set $A' \subseteq A$ that is close to A.

An Analogous Situation: Approximating Surface Area



$\mathbb{R}^n$

"Given a nice convex set such as a sphere, one can add a very thin tentacle to it with negligible volume but arbitrarily large surface area."

– Kothari, Nayyeri, O'Donnell, Wu (2014)

Refining The Original Question

Original Question: Given query access to $A \subseteq \mathbb{F}_2^n$ and writing

$$\mathrm{Vol}(A) := \frac{|A|}{2^n},$$

what is $\mathrm{Vol}(A+A)$ up to an error of $\pm\varepsilon$?

- Adding a small (random) collection $R \subseteq \mathbb{F}_2^n$ of $2^{0.51n}$ elements to A can blow up $\mathrm{Vol}(A+A)$ to almost 1.
- Natural relaxation: Output $\mathrm{Vol}(A'+A')$ for set $A' \subseteq A$ that is close to A.

The Question We Consider

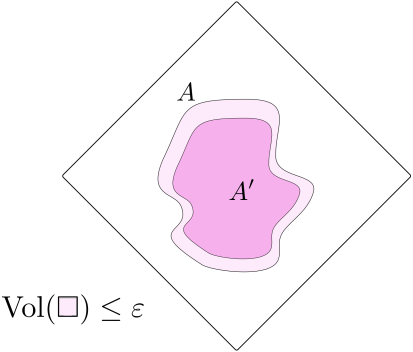
New Question: Given query access to $A \subseteq \mathbb{F}_2^n$ and writing

$$\mathrm{Vol}(A) := \frac{|A|}{2^n},$$

what is $\mathrm{Vol}\big(A' + A'\big)$ up to an error of $\pm\varepsilon$ for some $A' \subseteq A$ such that

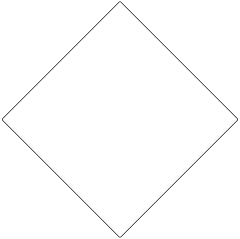
$$\mathrm{Vol}\big(A \setminus A'\big) \leq \varepsilon?$$

The Question We Consider



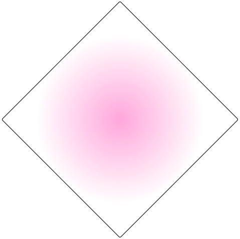
New Goal: Output $\mathrm{Vol}(A' + A')$ instead of $\mathrm{Vol}(A + A)$.

Revisiting Our Earlier Example



$A = \emptyset$

$\mathrm{Vol}(A + A) = 0$



$\boldsymbol{A}$ is a random set of size $2^{0.51n}$

$\mathrm{Vol}(\boldsymbol{A} + \boldsymbol{A}) \geq 1 - \exp(-n)$ w.h.p.

For $\varepsilon \geq 2^{-0.49n}$, simply output $\mathrm{Vol}(A' + A') = 0$.

Our Main Result

New Question: Given query access to $A \subseteq \mathbb{F}_2^n$ and writing

$$\mathrm{Vol}(A) := \frac{|A|}{2^n},$$

what is $\mathrm{Vol}\big(A' + A'\big)$ up to an error of $\pm\varepsilon$ for some $A' \subseteq A$ such that

$$\mathrm{Vol}\big(A \setminus A'\big) \leq \varepsilon?$$

Main Theorem: Can be done using $O_\varepsilon(1)$ queries to A.

(Bonus: Outputs an exact oracle to A' and an approximate oracle to $A' + A'$.)

Proof Sketch

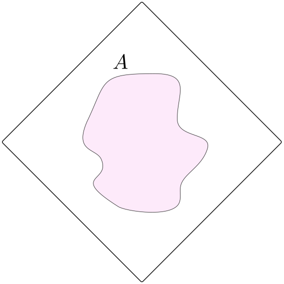
Almost all of $\mathbb{F}_2^n$

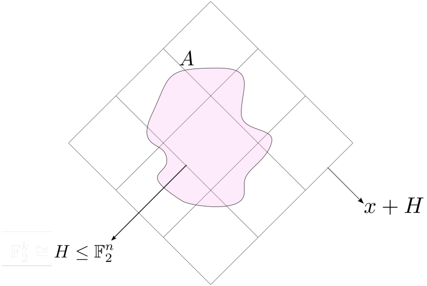
Ingredient 1: "Non-tiny" random-like sets have "large" sumsets.

Ingredient 2: Green's Regularity Lemma.

Need an algorithmic version

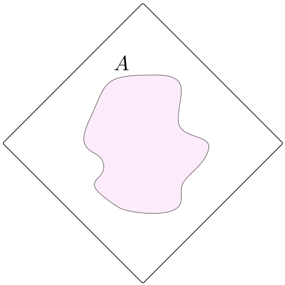
Green's Regularity Lemma

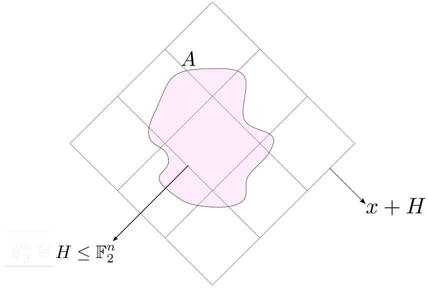




- Decomposes $\mathbb{F}_2^n$ into translates of $H \leq \mathbb{F}_2^n$ such that:
  - $H \cong \mathbb{F}_2^{n-k}$ where k is does not depend on n.
  - $A \cap (x + H)$ is "random-like," i.e. has small Fourier coefficients.
- Made algorithmic by closely following the original proof and using the Goldreich–Levin algorithm.

Sumset Simulation from 30,000 Feet





Defining A': Iterate through 2^k cosets of H:

- If $|A \cap (x + H)| \leq \varepsilon \cdot 2^{n-k}$, then set $A' \cap (x + H) = \emptyset$.
- Else set $A' \cap (x + H) = A$.

Approximately Defining $A' + A'$: If A' intersects with cosets $x + H, y + H$,

$$(A' + A') \ \cap \ (x + y + H) \approx x + y + H.$$

Ingredient 1

Obtaining $O_\varepsilon(1)$ Query Complexity

- Explicitly obtaining a description of the subspace H necessarily requires a number of queries that scales at least linearly in n.
- Require implicit versions of aforementioned algorithms.
  - For Goldreich–Levin: Equivalent to being a local list corrector for the Hadamard code.

Conclusion & Future Directions

- Our approach extends to estimate $\mathrm{Vol}(A + B)$ and $\mathrm{Vol}(A + \ldots + A)$ for $A, B \subseteq \mathbb{F}_2^n$.
- Generalizing to groups other than $\mathbb{F}_2^n$?
  - Green's Regularity Lemma does hold for arbitrary abelian groups.
  - Implicitly finding significant Fourier coefficients?

Thanks for listening! Questions?