

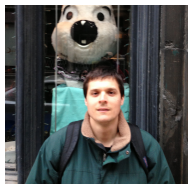
# Gaussian Polytope Approximation

Shivam Nadimpalli  
(MIT)

Joint work with



Anindya De  
(Penn)



Rocco Servedio  
(Columbia)

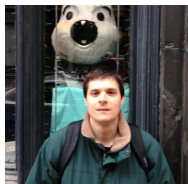
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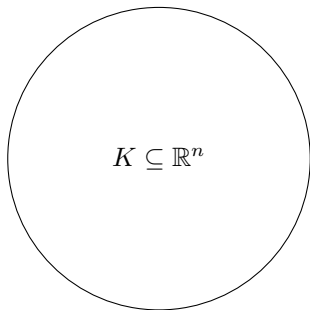


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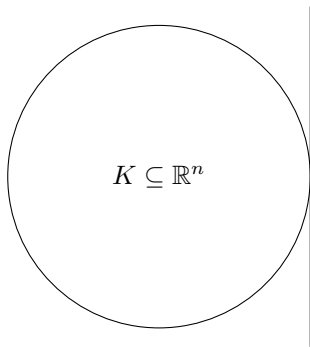


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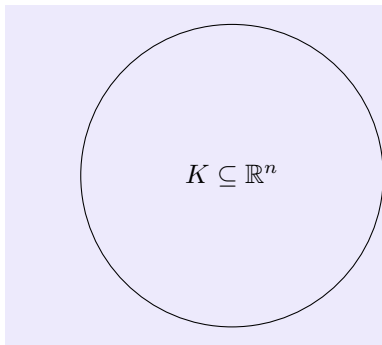
# Convexity 101



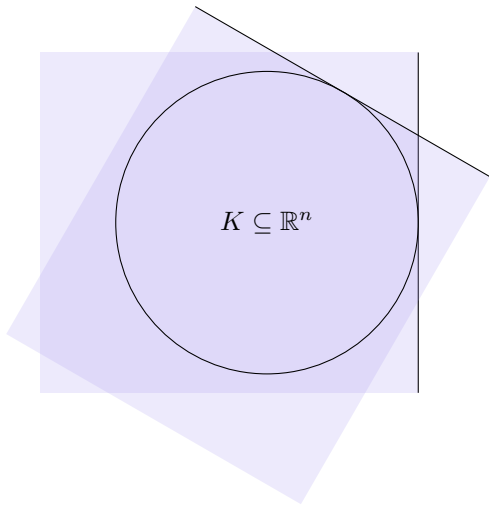
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**Fact:** Every convex set  $K \subseteq \mathbb{R}^n$  is an intersection of (possibly infinitely many) halfspaces, i.e.

$$K = \bigcap_i H_i$$

where  $H_i := \{x \in \mathbb{R}^n : \langle x, v^{(i)} \rangle \leq \theta_i\}$ .

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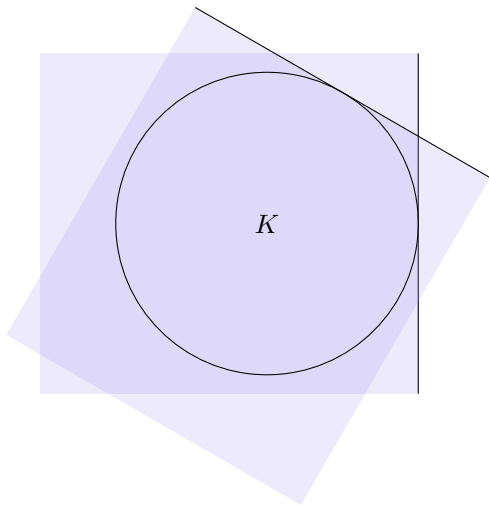
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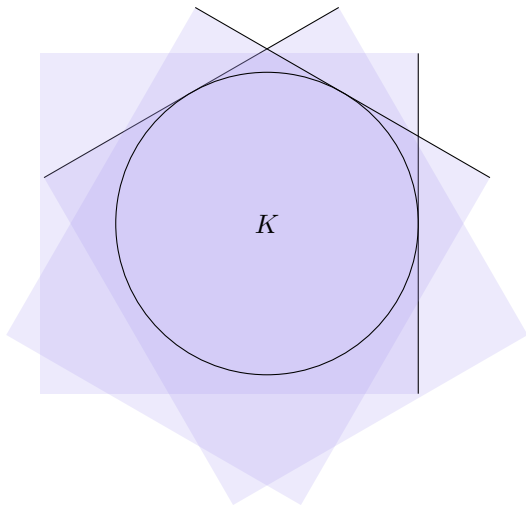
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What about **approximation**?

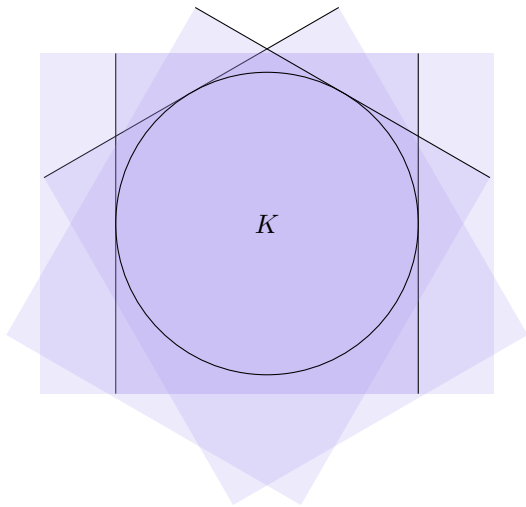
# Approximating Convex Sets



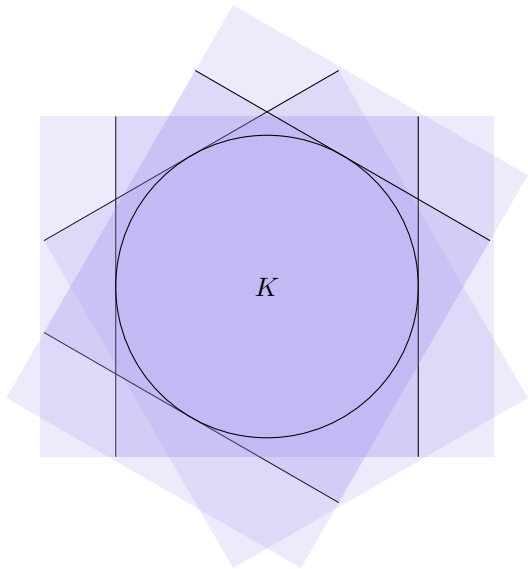
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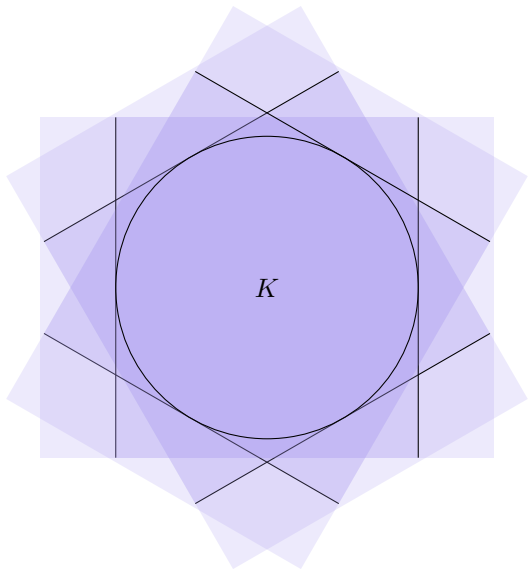
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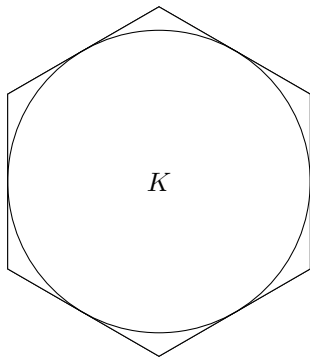
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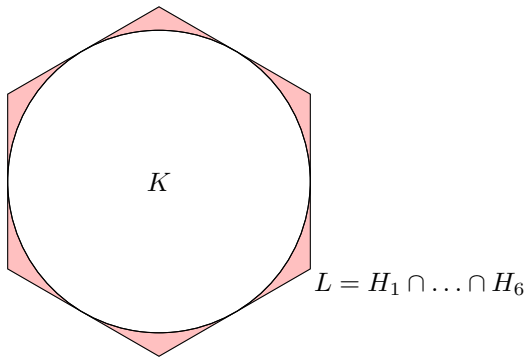


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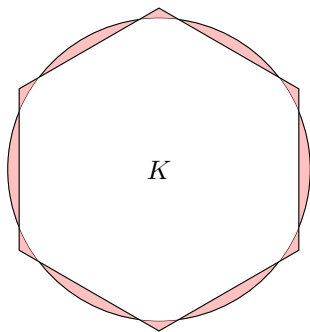


$$L = H_1 \cap \dots \cap H_6$$

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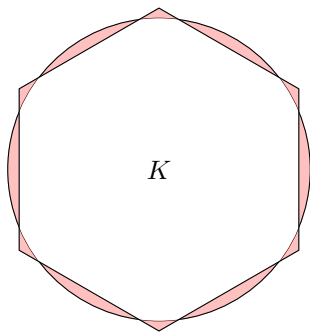


# Approximating Convex Sets



$$L' = H'_1 \cap \dots \cap H'_6$$

# Approximating Convex Sets



$$L' = H'_1 \cap \dots \cap H'_6$$

**Question:** How to measure “distance” between  $K$  and  $L, L'$ ?

# Gaussian Distance

**Definition:** For  $K, L \subseteq \mathbb{R}^n$ , we define the **Gaussian distance**

$$\text{dist}(K, L) := \Pr_{\mathbf{g} \sim N(0, I_n)} [\mathbf{g} \in K \triangle L]$$

where  $K \triangle L$  denotes the symmetric difference of  $K$  and  $L$ .

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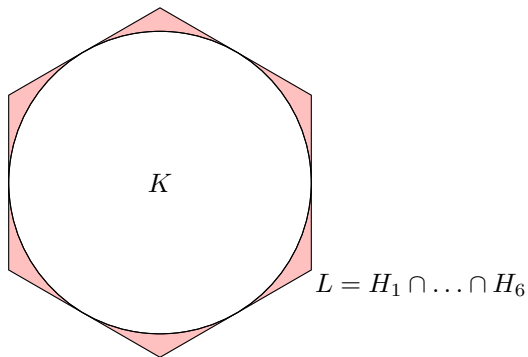
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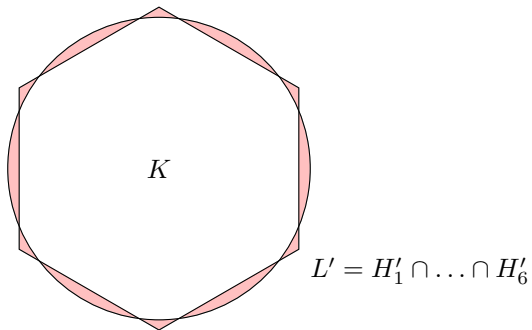
**Note:** Not translation invariant!

## Back to Approximation



$$\text{dist}(K, L) = \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right]$$

## Back to Approximation



$$\text{dist}(K, L') = \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right]$$

# Gaussian Polytope Approximation

**Question:** Given and  $0 < \varepsilon < 0.5$ , how many halfspaces are needed to  $\varepsilon$ -approximate a convex set  $K \subseteq \mathbb{R}^n$  under the Gaussian distance?

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**Want:** Upper bounds & lower bounds

# A Gaussian Twist on a Classical Problem

## APPROXIMATION BY POLYGONS AND POLYHEDRA

LÁSZLÓ FEJES TÓTH

In order to investigate as to what order of magnitude a plane (or a skew) curve can be approximated by  $n$ -sided polygons, we have to start from a definition of the *deviation*. One of the usual definitions is as follows:<sup>1</sup> the deviation  $\eta(H, K)$  of two curves  $H$  and  $K$  is the smallest number  $\epsilon$  for which  $H$  is contained in the neighbourhood of

# A Gaussian Twist on a Classical Problem

**APPROXIMATION BY POLYGONS AND POLYHEDRA**

In  
skew  
start  
is as  
small

JOURNAL OF APPROXIMATION THEORY **10**, 227–236 (1974)

**Metric Entropy of Some Classes of Sets with  
Differentiable Boundaries\***

R. M. DUDLEY

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JOURNAL OF APPROXIMATION THEORY 10, 227-236 (1974)

In  
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result

### THE APPROXIMATION OF CONVEX SETS BY POLYHEDRA

E. M. Bronshteyn and L. D. Ivanov

The main result of this note is the following theorem.

**THEOREM.** Let  $M$  be a convex subset of the unit ball in  $n$ -dimensional Euclidean space. Then for any  $\varepsilon \in (0, 10^{-3})$  there are no more than  $3\sqrt{n} \cdot (\pi/\varepsilon)^{(n-1)/2}$  points whose convex hull is  $\varepsilon$  away from  $M$  in the Hausdorff metric.

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PROCEEDINGS OF THE  
AMERICAN MATHEMATICAL SOCIETY  
Volume 102, Number 3, March 1988

## APPROXIMATION OF THE SPHERE BY POLYTOPES HAVING FEW VERTICES

I. BÁRÁNY AND Z. FÜREDI

(Communicated by Andrew Odlyzko)

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*Journal of Mathematical Sciences, Vol. 153, No. 6, 2008*

## APPROXIMATION OF CONVEX SETS BY POLYTOPES

**E. M. Bronstein**

ABSTRACT. The survey contains results related to different aspects of polyhedral convex bodies and some adjacent problems.

# Algorithms, Gaussians, and Convexity

## Learning Convex Sets [KOS'08]

Output an  $\varepsilon$ -approximation to a convex set  $K$

## Testing Convexity [CFSS'17]

Output “yes” if  $K$  convex and “no” if  $\varepsilon$ -far from convex

## Estimation from Convex-Truncated Gaussians [DGTZ'18]

Estimate  $\mu$  and  $\Sigma$  given samples from  $N(\mu, \Sigma) \upharpoonright_K$

## Detecting Convex Truncation [DNS'23]

Distinguish  $N(0, I_n)$  from  $N(0, I_n) \upharpoonright_K$  for  $K$  convex

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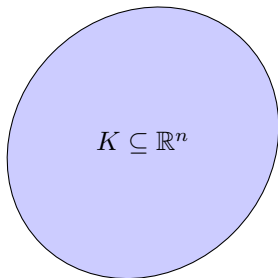
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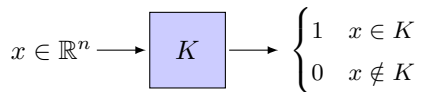
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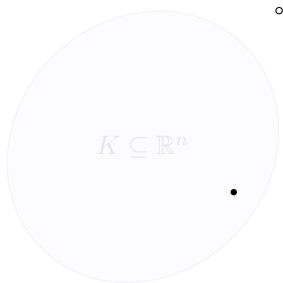


# Learning Convex Sets



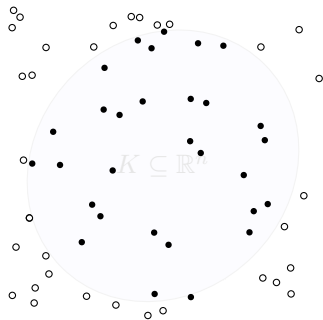
$$x \in \mathbb{R}^n \rightarrow \boxed{K} \rightarrow \begin{cases} 1 & x \in K \\ 0 & x \notin K \end{cases}$$

# Learning Convex Sets



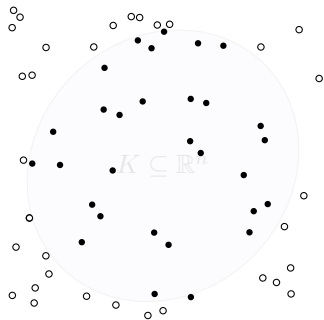
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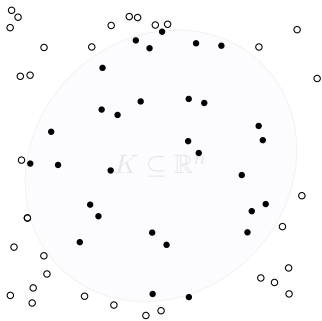
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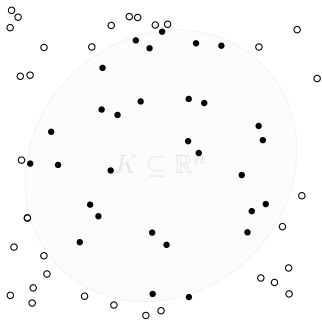
**Goal:** Output approximation to  $K$

# Learning Convex Sets



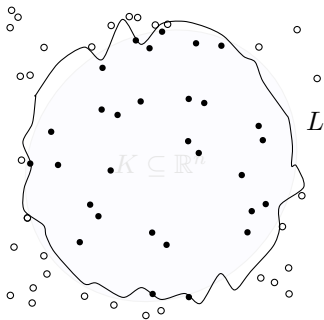
0.1-approximation to  $K$ :

# Learning Convex Sets



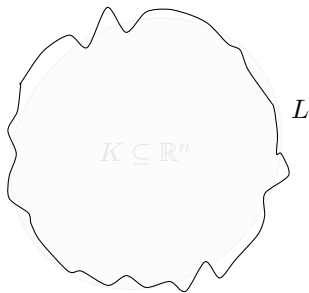
0.1-approximation to  $K$ :  $n^{\Theta(\sqrt{n})}$  queries [KOS'08]

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# Learning Convex Sets



0.1-approximation to  $K$ :  $n^{\Theta(\sqrt{n})}$  queries [KOS'08]

But **improper** ☹️

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**Question:** Given a convex set  $K \subseteq \mathbb{R}^n$ , how many halfspaces are needed to 0.1-approximate  $K$  under the Gaussian distribution?

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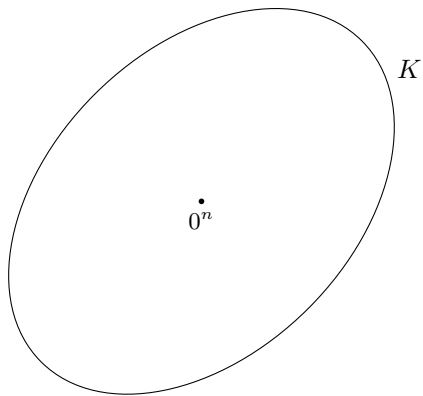
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# An Exponential Upper Bound

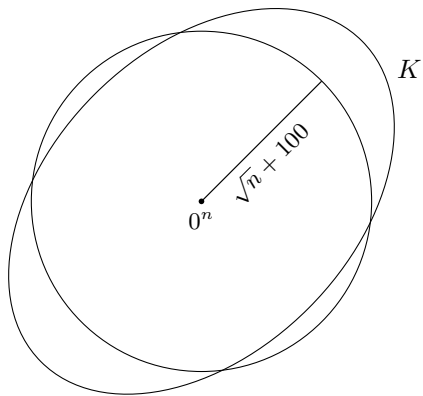
**Theorem:** For every convex  $K \subseteq \mathbb{R}^n$ , there exists  $L \subseteq \mathbb{R}^n$  which is an intersection of  $2^{\tilde{O}(n)}$  halfspaces such that

$$\text{dist}(K, L) \leq 0.1.$$

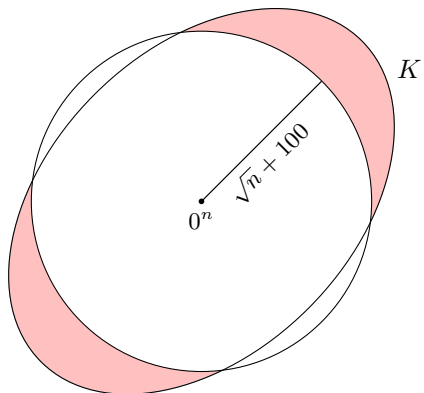
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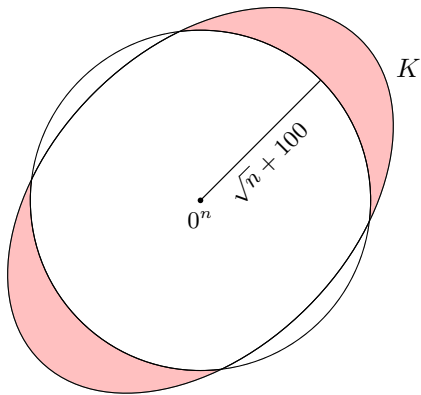
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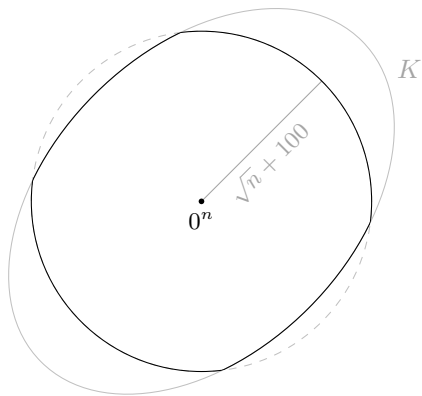


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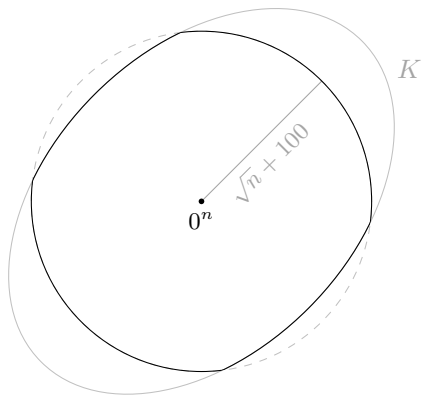


$\chi(n)$ -Tail Bound:  $\Pr_{\mathbf{g} \sim N(0, I_n)} [\|\mathbf{g}\|_2 \geq \sqrt{n} + 100] \leq 0.05.$

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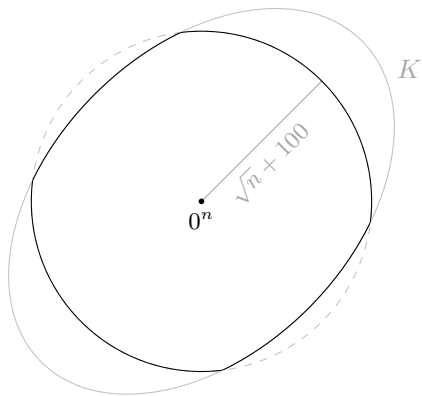


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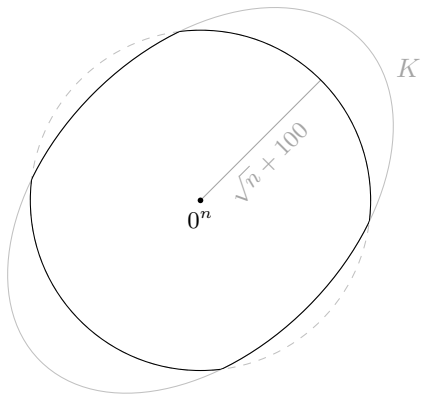
**New Goal:** 0.05-approximation to  $K \cap B_2(\sqrt{n} + 100) =: K'$ .

## An Exponential Upper Bound



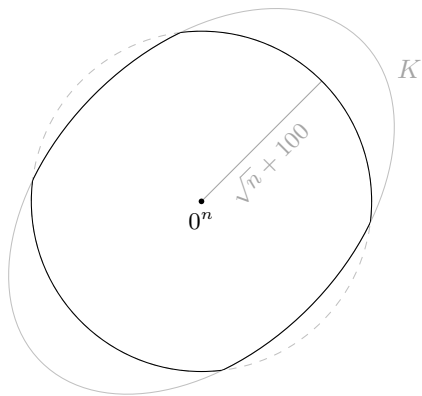
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# Hausdorff Approximation to the Rescue

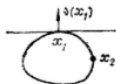


Fig. 1

[Dud'74, BI'75]: There exists a  $2^{\tilde{O}(n)}$ -facet polytope  $L$  such that  $K' \subseteq L$  and

$$\sup_{y \in L} \|y - K'\| \leq \frac{0.05}{n^{1/4}}.$$

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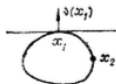
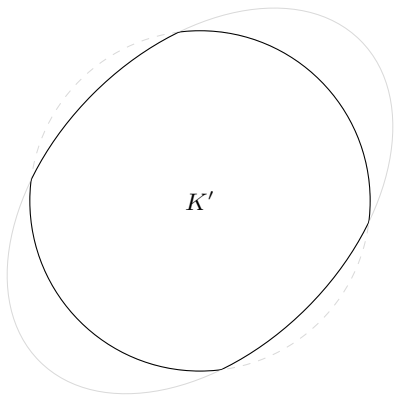


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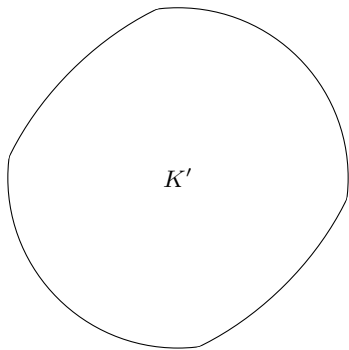
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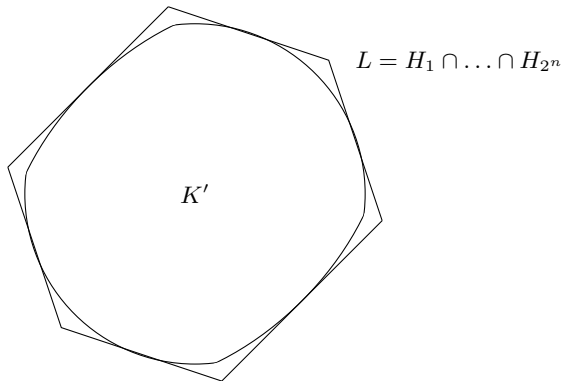
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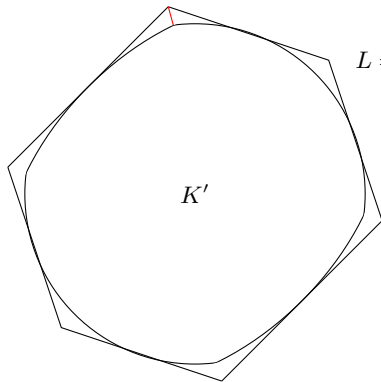


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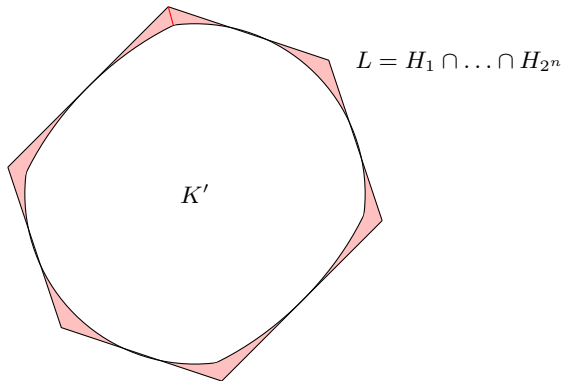
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$$\text{length}(-) \leq 0.05n^{-1/4}$$



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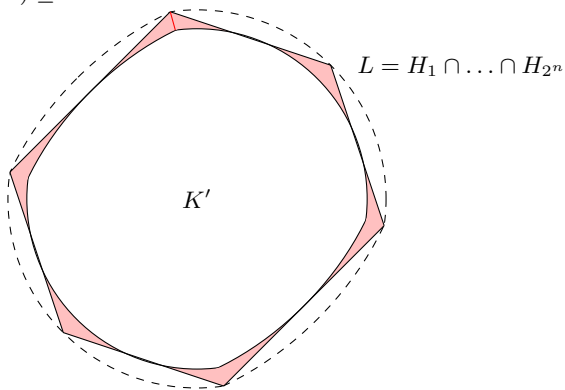
$$\text{length}(-) \leq 0.05n^{-1/4}$$



$$\text{dist}(K', L) = \Pr_{\mathbf{g} \sim N(0, I_n)} [\mathbf{g} \in \text{red}]$$

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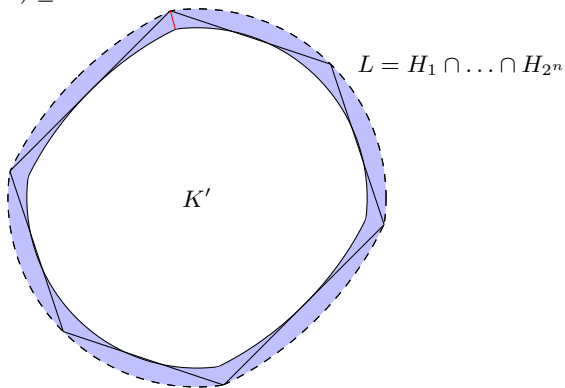
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# An Exponential Upper Bound

$$\text{length}(-) \leq 0.05n^{-1/4}$$



$$\text{dist}(K', L) = \Pr_{g \sim N(0, I_n)} [g \in \text{red}] \leq \Pr_{g \sim N(0, I_n)} [g \in \text{blue}]$$

## Some Easy Calculus

$$\begin{aligned} \text{dist}(K', L) &= \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right] \\ &\leq \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right] \end{aligned}$$

## Some Easy Calculus

$$\text{dist}(K', L) = \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right]$$

$$\leq \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right]$$

$$= \int_{\{x : \|x - K'\| \leq 0.05n^{-1/4}\}} \varphi_n(x) dx$$

$\varphi_n =$  Gaussian pdf

## Some Easy Calculus

$$\begin{aligned} \text{dist}(K', L) &= \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right] \\ &\leq \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right] \\ &= \int_{\{x : \|x - K'\| \leq 0.05n^{-1/4}\}} \varphi_n(x) dx && \varphi_n = \text{Gaussian pdf} \\ &= \int_{t=0}^{0.05n^{-1/4}} \int_{x \in \partial K'_t} \varphi_n(x) d\sigma(x) dt && K'_t := \{x : \|x - K'\| \leq t\} \end{aligned}$$

## Some Easy Calculus

$$\begin{aligned} \text{dist}(K', L) &= \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \text{red square} \right] \\ &\leq \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \text{blue square} \right] \\ &= \int_{\{x : \|x - K'\| \leq 0.05n^{-1/4}\}} \varphi_n(x) dx && \varphi_n = \text{Gaussian pdf} \\ &= \int_{t=0}^{0.05n^{-1/4}} \int_{x \in \partial K'_t} \varphi_n(x) d\sigma(x) dt && K'_t := \{x : \|x - K'\| \leq t\} \\ &= \int_{t=0}^{0.05n^{-1/4}} \text{GSA}(K'_t) dt \end{aligned}$$

# Gaussian Surface Area of Convex Sets



[Ball'93]: Suppose  $K \subseteq \mathbb{R}^n$  is a convex set. Then

$$\text{GSA}(K) \leq n^{1/4}$$

where  $\text{GSA}(K) = \int_{\partial K} \varphi_n(x) d\sigma(x)$ .

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# Putting Everything Together

**Theorem:** For every convex  $K \subseteq \mathbb{R}^n$ , there exists  $L \subseteq \mathbb{R}^n$  which is an intersection of  $2^{\tilde{O}(n)}$  halfspaces such that

$$\text{dist}(K, L) \leq 0.1.$$

Key ideas: Truncation + [Dud'74, Bl'75] + [Ball'93]

# Putting Everything Together

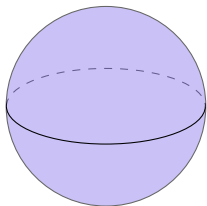
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Key ideas: Truncation + [Dud'74, Bl'75] + [Ball'93]

What about **lower bounds**?

## A Natural Guess

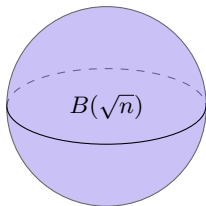


$$B_2(\sqrt{n}) := \{x \in \mathbb{R}^n : \|x\| \leq \sqrt{n}\}$$

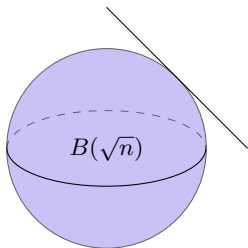
Extremal for Lebesgue, Hausdorff, Banach–Mazur, ...

## Warmup: “Outer” Approximators

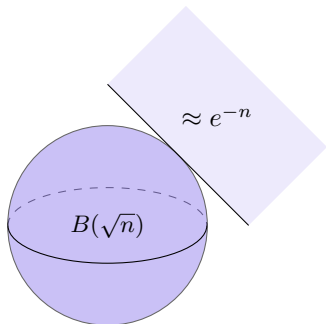
## Warmup: “Outer” Approximators



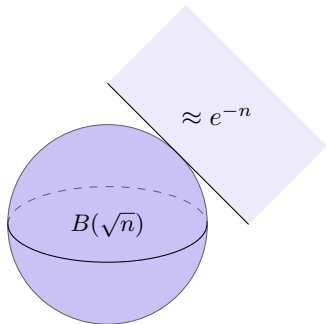
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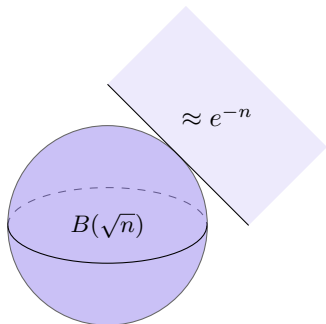


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Union bound  $\implies 2^{\Omega(n)}$  halfspaces for outer approximation 😊

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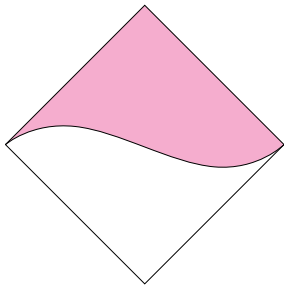


Union bound  $\implies 2^{\Omega(n)}$  halfspaces for outer approximation 😊

What about **general approximators**?



## A Brief Aside



Monotone  $f : \{0, 1\}^n \rightarrow \{0, 1\}$

(If  $x \preceq y$  then  $f(x) \leq f(y)$ )

# Monotone Boolean Functions

Analytic Structure    Fourier concentration up to degree  $O(\sqrt{n})$

Density Increment Results    Kruskal–Katona

Uniform Distribution Learning     $n^{O(\sqrt{n})}$  time for constant accuracy

Structural Results    Harris–Kleitman, KKL, Talagrand, and Friends

Testing Distribution Truncation     $\Theta(n)$  samples

# Monotone Boolean Functions & Convex Sets

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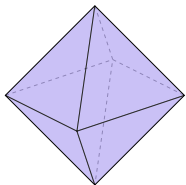
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A Kruskal–Katona Analogue

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 $N(0, I_n)$      $n^{O(\sqrt{n})}$  time for constant accuracy

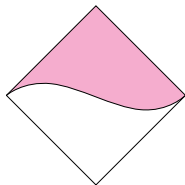
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Royen, Convex Influences [DNS'21, DNS'22]

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# An Emerging Analogy

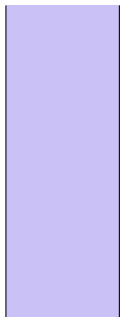


Convex Subsets of  $\mathbb{R}^n$



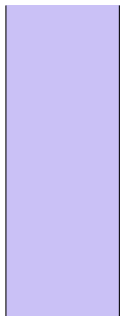
Monotone Subsets of  $\{0, 1\}^n$

# Slabs



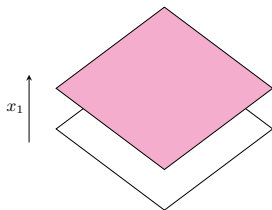
$$S := \{x \in \mathbb{R}^n : |x_1| \leq 1\}$$

# Slabs are like Boolean Dictatorships



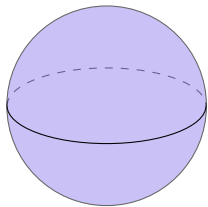
$\xrightarrow{e_1}$

$$S := \{x \in \mathbb{R}^n : |x_1| \leq 1\}$$



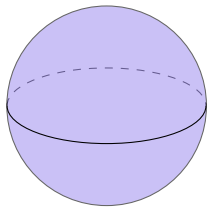
$$\text{Dict}_1(x) := x_1$$

# The $\ell_2$ -Ball

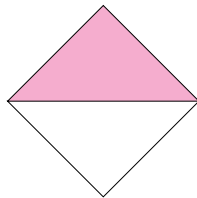


$$B_2 := \{x \in \mathbb{R}^n : \|x\| \leq \sqrt{n}\}$$

# The $\ell_2$ -Ball is like Majority

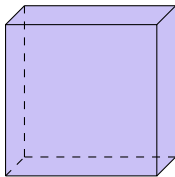


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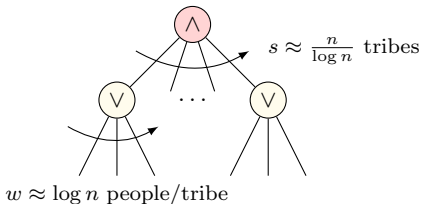
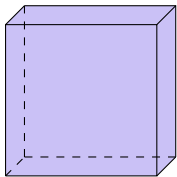
$$\text{Maj}_n(x) := \mathbf{1}(\sum_{i=1}^n x_i \geq n/2)$$

# The $\ell_\infty$ -Ball



$$C := \left\{ x \in \mathbb{R}^n : |x|_\infty \leq \Theta(\sqrt{\log n}) \right\}$$

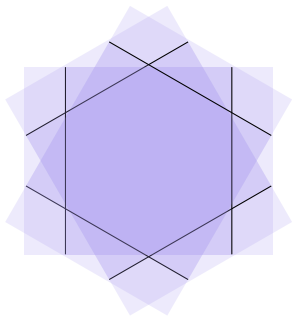
The  $\ell_\infty$ -Ball is like Tribes



$$C := \left\{ x \in \mathbb{R}^n : |x|_\infty \leq \Theta(\sqrt{\log n}) \right\}$$

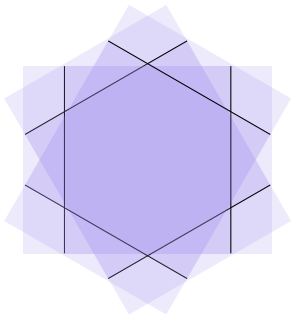
$$\text{Tribes}_n(x) = \bigwedge_{i=1}^s \left( \bigvee_{j=1}^w x_{i,j} \right)$$

# An Intersection of Halfspaces

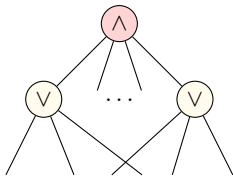


$$K = \bigcap_i H_i$$

# An Intersection of Halfspaces is like a CNF

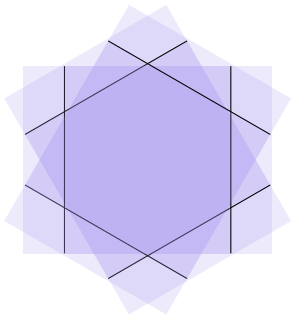


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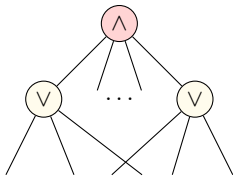


$$f(x) = \bigwedge_i \left( \bigvee_j x_{ij} \right)$$

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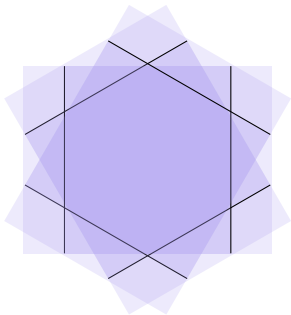
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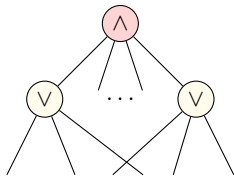
Clause

# An Intersection of Halfspaces is like a CNF



$$K = \bigcap_i H_i$$

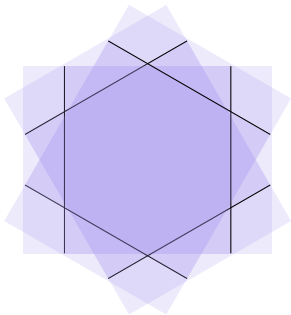
Polytopal Approximation



$$f(x) = \bigwedge_i \left( \bigvee_j x_{ij} \right)$$

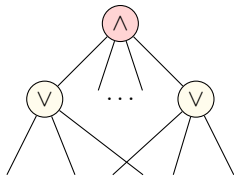
Approximation by CNF

# An Intersection of Halfspaces is like a CNF



$$K = \bigcap_i H_i$$

Polytopal Approximation



$$f(x) = \bigwedge_i \left( \bigvee_j x_{ij} \right)$$

Approximation by CNF

## Approximation by CNF

**Question:** Given a Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ , how many clauses are needed by a CNF  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  such that

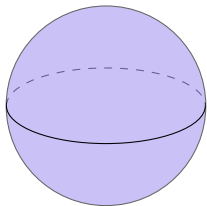
$$\Pr_{\mathbf{x} \sim \{0,1\}^n} [f(\mathbf{x}) \neq g(\mathbf{x})] \leq 0.1?$$

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## Back to Polytope Approximation

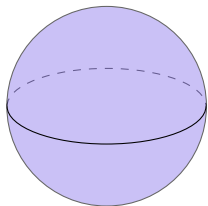


$$B_2(\sqrt{n}) := \{x \in \mathbb{R}^n : \|x\| \leq \sqrt{n}\}$$

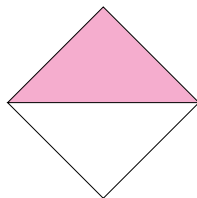
Extremal for Lebesgue, Hausdorff, Banach–Mazur, ...



## Consulting the Analogy

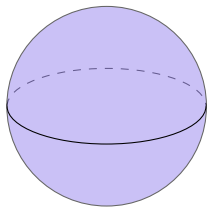


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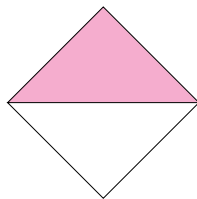


$$\text{Maj}_n(x) := \mathbf{1}(\sum_{i=1}^n x_i \geq n/2)$$

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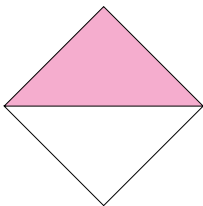
# Total Influence of a Boolean Function

**Definition:** The **total influence** of a Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  is defined as

$$\mathbf{I}[f] := \mathbf{E}_{\mathbf{x} \sim \{0,1\}^n} \left[ \#\left\{i : f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus i})\right\} \right]$$

where  $\mathbf{x}^{\oplus i} = (x_1, \dots, 1 - x_i, \dots, x_n)$ .

# The Total Influence of Majority

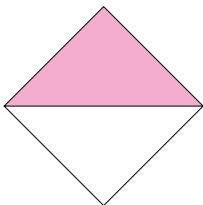


$$\text{Maj}_n(x) := \mathbf{1}(\sum_{i=1}^n x_i \geq n/2)$$

Question: What is  $\mathbf{I}[\text{Maj}_n]$ ?

$$\text{Recall } \mathbf{I}[f] := \mathbf{E} \left[ \#\{i : f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus i})\} \right]$$

# The Total Influence of Majority



$$\text{Maj}_n(x) := \mathbf{1}(\sum_{i=1}^n x_i \geq n/2)$$

Question: What is  $\mathbf{I}[\text{Maj}_n]$ ?

Answer:  $\Theta(\sqrt{n})$

Recall  $\mathbf{I}[f] := \mathbf{E} \left[ \#\{i : f(x) \neq f(x^{\oplus i})\} \right]$

# The Total Influence of CNFs



[Håstad–Boppana'97]: Any  $s$ -clause CNF  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  has

$$\mathbf{I}[g] \leq O(\log s).$$

# The Total Influence of CNFs



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# The Total Influence of CNFs



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$$\mathbf{I}[\text{Maj}_n] = \Omega(\sqrt{n}) \implies \text{0.1-approx. needs } 2^{\Omega(\sqrt{n})} \text{ clauses}$$

(Fourier + monotonicity behind the scenes)

# Monotone Boolean Functions & Convex Sets

Analytic Structure    Fourier concentration up to degree  $O(\sqrt{n})$   
Fourier Hermite concentration up to degree  $O(\sqrt{n})$

Density Increment Results    Kruskal–Katona  
A Kruskal–Katona Analogue

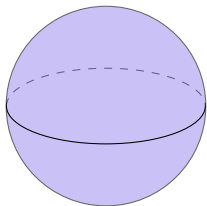
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 $\Theta(n)$  samples [DNS'23]

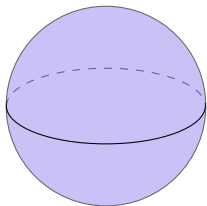


## Back to Polytope Approximation



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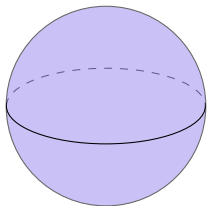
## Back to Polytope Approximation



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Can follow similar recipe!

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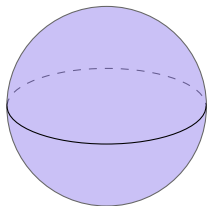
Convex Influences + Håstad–Boppana Analogue

(Fourier + convexity behind the scenes)

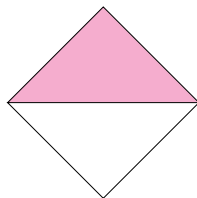
## Story So Far...

	Upper Bound	Lower Bound
Generic $K \subseteq \mathbb{R}^n$	$2^{\tilde{O}(n)}$	—
$\ell_2$ -Ball	—	$2^{\Omega(\sqrt{n})}$

## Consulting the Analogy Once Again

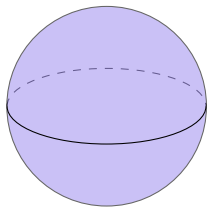


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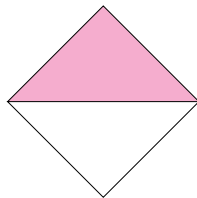


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## Approximation by DNF: Examples and Counterexamples

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May 1, 2007

# Approximation by DNF: Examples and Counterexamples

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## Abstract

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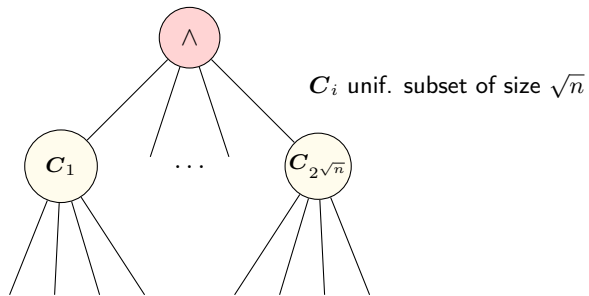
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# A Matching Upper Bound

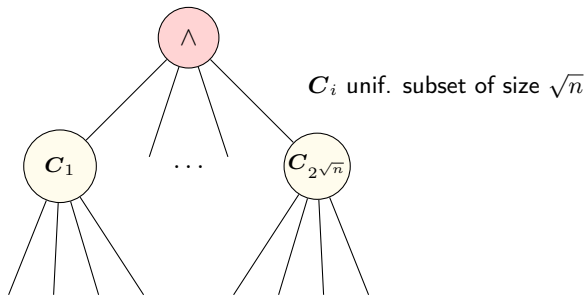


[OW'07]: There exists a  $2^{O(\sqrt{n})}$ -clause CNF that is a 0.1-approximator to  $\text{Maj}_n$ .

# The O'Donnell–Wimmer Construction

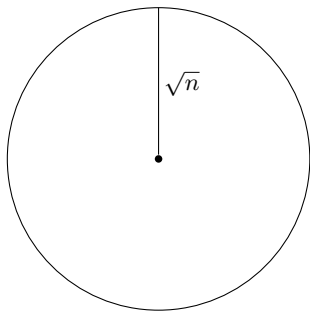


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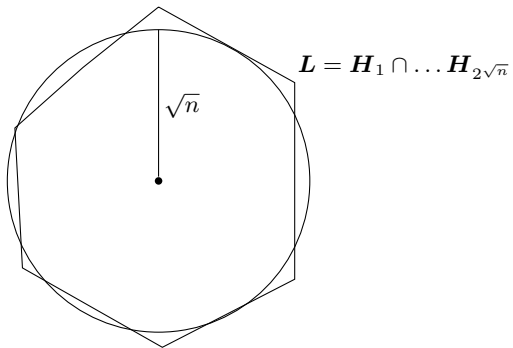


Key Idea:  $\mathbf{E}_f \left[ \Pr_{\mathbf{x} \sim \{0,1\}^n} [\mathbf{f}(\mathbf{x}) \neq \text{Maj}_n(\mathbf{x})] \right] \leq 0.1$

# Nazarov's Body

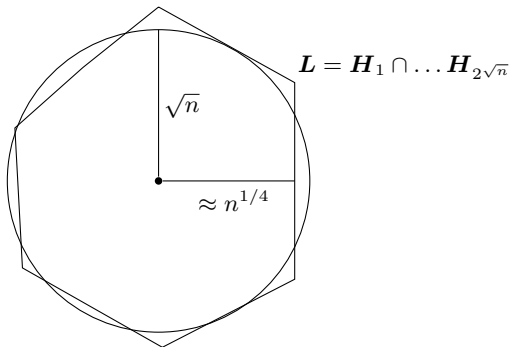


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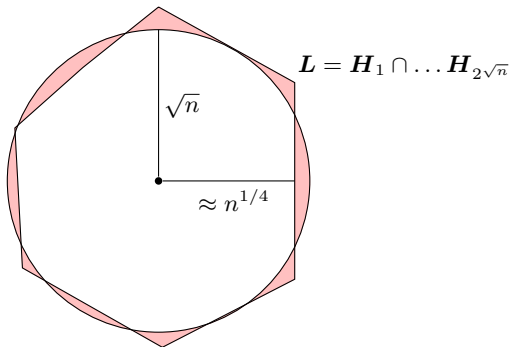
$$H_i = \{x : \langle x, g^{(i)} \rangle \leq n^{3/4}\}$$

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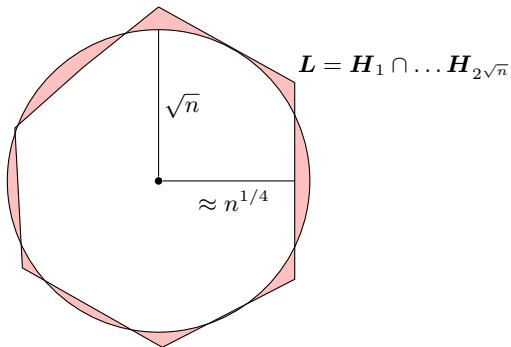
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Similar Idea:  $\mathbf{E} \left[ \Pr_{\mathbf{g} \sim N(0, I_n)} \left[ \mathbf{g} \in \square \right] \right] \leq 0.1$

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## Story So Far...

	Upper Bound	Lower Bound
Generic $K \subseteq \mathbb{R}^n$	$2^{\tilde{O}(n)}$	—
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# Approximation by DNF: Examples and Counterexamples

Ryan O'Donnell  
Carnegie Mellon University

Karl Wimmer  
Carnegie Mellon University

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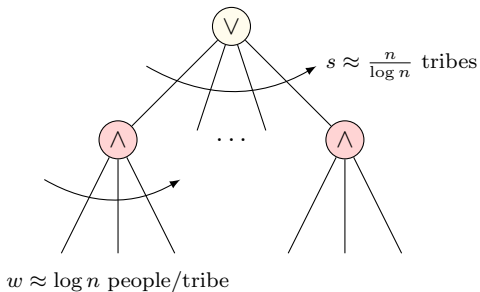
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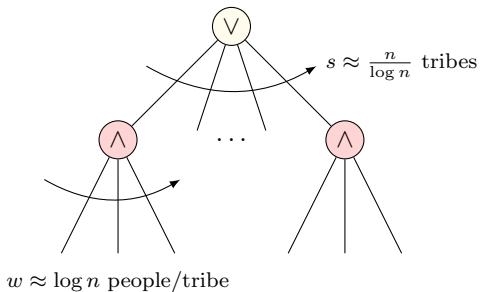
[OW'07]: There exists a monotone Boolean function such that any CNF that is a 0.1-approximator must have  $2^{\tilde{\Omega}(n)}$  clauses.

Key idea (in hindsight): Random “projections” [RST'15]

# O'Donnell & Wimmer's Hard Function

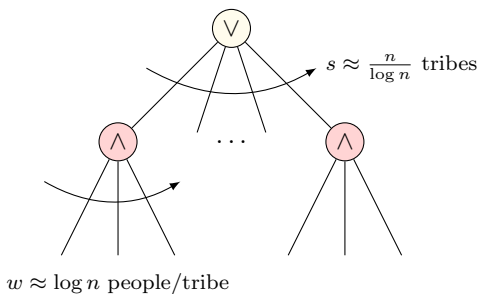


# O'Donnell & Wimmer's Hard Function



**DNF** Tribes

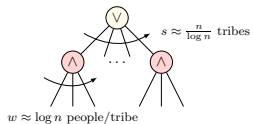
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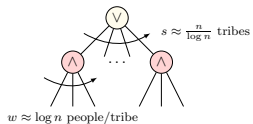
Question: What's the convex analogue?

# Following the Analogy

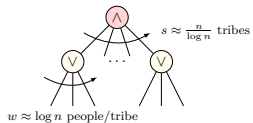


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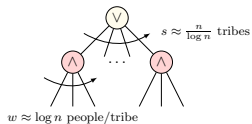


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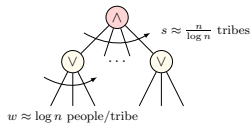
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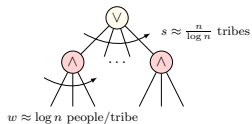
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↑  
Boolean Dual  
↓



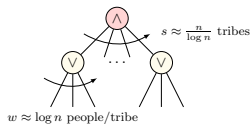
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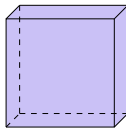
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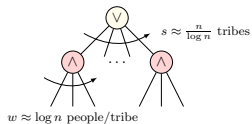
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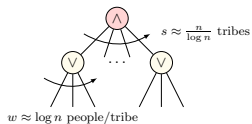
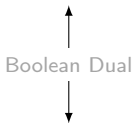


$l_\infty$ -ball

# Following the Analogy



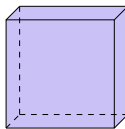
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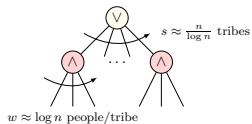
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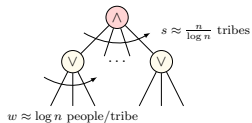
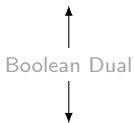


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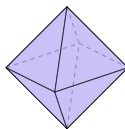


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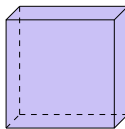


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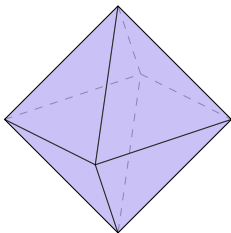


$\ell_1$ -ball



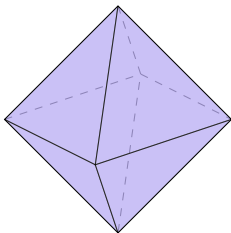
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## A Reasonable Guess



**Conjecture:** Any 0.1-approximator to the  $\ell_1$ -ball of Gaussian measure 0.5 must have  $2^{\Omega(n)}$  halfspaces.

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**False!** There exists a sub-exponential approximator 😊

# A Proof Sketch

**Probabilistic construction**

Distinct from  $\ell_2$ -ball approximator

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## Probabilistic construction

Distinct from  $\ell_2$ -ball approximator

## Sample dilated $\ell_1$ -balls in subspaces

Take cylinders of these “juntas”

## Anti-concentration & multiplicative Gaussian tail bounds [Pet'12]

Standard trickery (Gaussian tails, Berry-Esseen) fail

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**There's More:** Average-Case Lower Bounds...

Connections to learning and pseudorandomness

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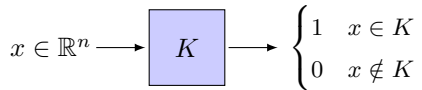
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# Convexity Testing



**Question:** How many queries to  $K \subseteq \mathbb{R}^n$  to distinguish w.p.  $2/3$  between the following?

- $K$  is convex, vs.
- $\text{dist}(K, L) \geq 0.1$  for every convex  $L \subseteq \mathbb{R}^n$ .

# Convexity Testing



Queries

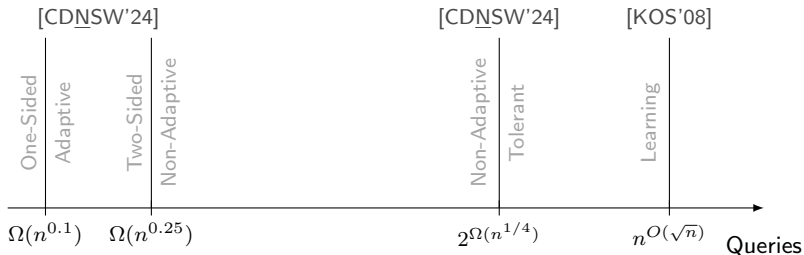
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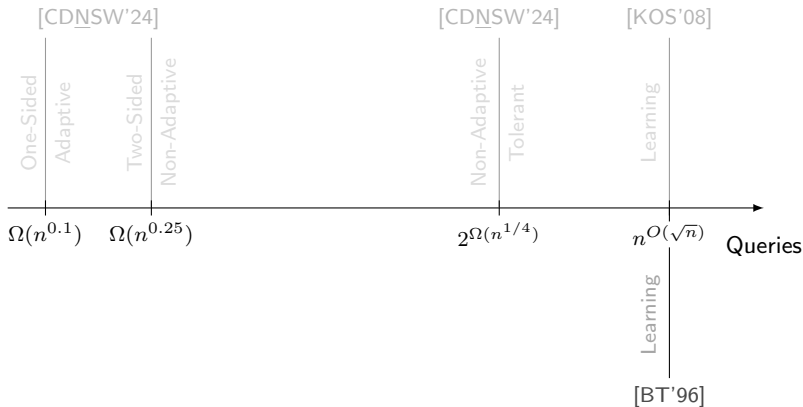


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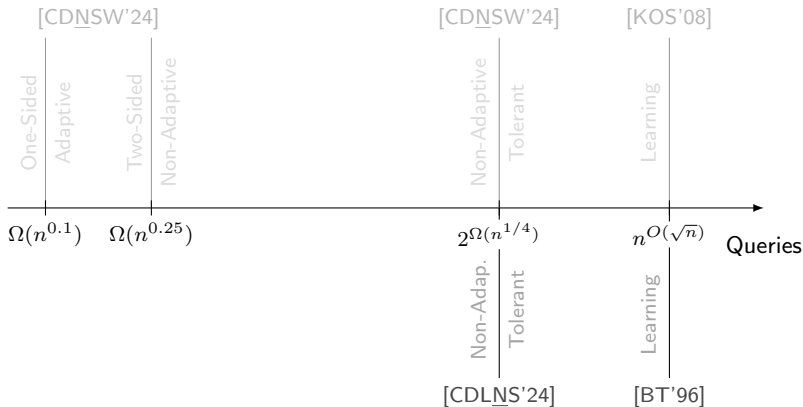
(Tolerant testing  $\approx$  estimating distance to property)

# Convexity Testing vs. Monotonicity Testing



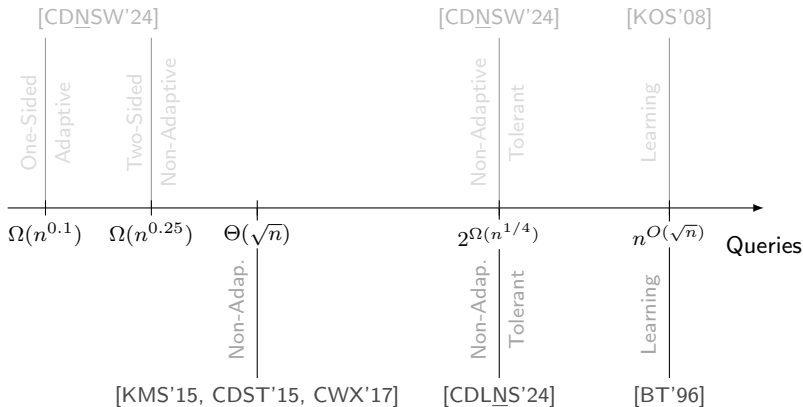
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Thanks for listening! Questions?