RATIONALITY OF TWISTS OF $\mathcal{A}(3)$

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Abstract. Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be a continuous Galois representation with cyclotomic similitude character — or, what turns out to be equivalent, the Galois representation associated to the 3-torsion of a principally polarized abelian surface $A/\mathbf{Q}$. We prove that the moduli space $\mathcal{A}_2(\overline{\rho})$ of principally polarized abelian surfaces $B/\mathbf{Q}$ admitting a symplectic isomorphism $B[3] \simeq \overline{\rho}$ of Galois representations is never rational over $\mathbf{Q}$ when $\overline{\rho}$ is surjective, even though it is both rational over $\mathbf{C}$ and unirational over $\mathbf{Q}$ via a map of degree 6.

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1. Introduction

Suppose that $A/\mathbf{Q}$ is an abelian variety of dimension g with a polarization of degree prime to p. Associated to the action of the absolute Galois group $G_{\mathbf{Q}}$ on $A[p]$ there exists a Galois representation

$$\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_{2g}(\mathbf{F}_p)$$

such that the corresponding similitude character is the mod-p cyclotomic character ε. One can ask, conversely, whether any such representation come from an abelian surface in infinitely many ways. When $g = 1$, this question is well studied, and has a positive answer exactly for $p = 2$, 3, and 5. This corresponds to the fact that the corresponding twists $X(\overline{\rho})$ of the modular curve $X(p)$ are rational over $\mathbf{Q}$ for $p = 2$, 3, and 5 respectively, and have higher genus for larger p.

In [BCGP18], the corresponding question arose for abelian surfaces when $p = 3$. (The case $p = 2$, which is also discussed in that paper, is easy.) Let $\mathcal{A}_2(3)$ denote the moduli space of principally polarized abelian surfaces together with a symplectic isomorphism $A[3] \simeq (\mathbf{Z}/3\mathbf{Z})^2 \oplus (\mu_3)^2$. Given a $\overline{\rho}$ as above, one can form the corresponding moduli space $\mathcal{A}_2(\overline{\rho})$ where now one insists that there is a symplectic isomorphism $A[3] \simeq V$, where V is the symplectic representation associated to $\overline{\rho}$. The variety $\mathcal{A}_2(\overline{\rho})$ is well known to be a smooth open subvariety of the Burkhardt quartic, which is rational over $\mathbf{Q}$ ([BN18]). It is clear that $\mathcal{A}_2(\overline{\rho})$ is isomorphic

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to $\mathcal{A}_2(3)$ over $\mathbf{C}$ (and even over the fixed field of the kernel of $\overline{\rho}$), and hence $\mathcal{A}_2(\overline{\rho})$ is *geometrically* rational. If $\mathcal{A}_2(\overline{\rho})$ was in fact *rational* (by which we always mean rational over the corresponding base field), then indeed the answer to the question above would be positive, just as for elliptic curves when $p \le 5$. In [BCGP18, Prop 10.2.3] (see also [CCR20]), the following weaker result was established.

Theorem 1.1. *The variety $\mathcal{A}_2(\overline{\rho})$ is unirational over $\mathbf{Q}$ via a map of degree at most 6.*

As a consequence, any such $\overline{\rho}$ *does* arise from (infinitely many) abelian surfaces. However, the question as to whether $\mathcal{A}_2(\overline{\rho})$ was actually rational was left open. We address this question here. A special case of our main result is the following:

Theorem 1.2. *Suppose that $\overline{\rho}$ is surjective. Then $\mathcal{A}_2(\overline{\rho})$ is not rational, and the minimal degree of any rational cover is 6.*

Clearly (in light of Theorem 1.1) the constant 6 is best possible in this case. Note that the answer may (in general) depend on $\overline{\rho}$, since the case when $\overline{\rho}$ is trivial corresponds to $\mathcal{A}_2(3)$, which is rational. In addition to Theorem 1.2, we also prove the following:

Theorem 1.3. *Let $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GSp}_4(\mathbf{F}_3)$ be a representation with cyclotomic similitude character. Suppose that the order of $\mathrm{im}(\overline{\rho})$ has order greater than 96. Then $\mathcal{A}_2(\overline{\rho})$ is not rational over $\mathbf{Q}$.*

More precise results can be extracted directly from the table in §3. By assumption, the image of $\overline{\rho}$ restricted to G_E with $E = \mathbf{Q}(\sqrt{-3})$ lands inside $\mathrm{Sp}_4(\mathbf{F}_3)$. If we let H denote the projection of $\mathrm{im}(\overline{\rho}|_{G_E})$ to the simple group $\mathrm{PSp}_4(\mathbf{F}_3)$, then we prove that $\mathcal{A}_2(\overline{\rho})$ is not rational over $\mathbf{Q}$ for all but 27 of the 116 conjugacy classes of subgroups of $\mathrm{PSp}_4(\mathbf{F}_3)$. With the exception of the case when H is trivial, we do not know what happens in the remaining 26 cases, nor do we even know whether the rationality of $\mathcal{A}_2(\overline{\rho})$ depends only on $\mathrm{im}(\overline{\rho})$ or not.

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2. The Strategy

The main idea behind the proof is to follow a strategy employed by Manin for cubic surfaces. Recall (§A.1 of [Man86]) that a continuous G_K-module with the discrete topology is called a *permutation module* if it admits a finite free $\mathbf{Z}$-basis on which G_K acts (via a finite quotient) via permutations, and that two modules M and N are *similar* if $M \oplus P \simeq N \oplus Q$ for permutation modules P and Q. In particular, we employ the following theorem:

Theorem 2.1. [Man86, §A.1 Theorem 2] *Let Z be a smooth projective algebraic variety over a number field K. Suppose that Z is rational over K. Then $\mathrm{Pic}_{\overline{K}} Z$ as a G_K-representation is similar to the zero representation.*

The Shimura variety $\mathcal{A}_2(3)$ admits a smooth toroidal projective compactification $\mathcal{A}_2^*(3)$, the (canonical) toroidal compactification constructed by Igusa. The automorphism group of $\mathcal{A}_2^*(3)$ over $\overline{\mathbf{Q}}$ is the group $G = \mathrm{PSp}_4(\mathbf{F}_3)$, the simple group of order 25920, which acts over the field $E = \mathbf{Q}(\sqrt{-3})$. It will be convenient from this point onwards to always work over the field E. (Certainly rationality over $\mathbf{Q}$ implies rationality over E, so non-rationality over E implies non-rationality over $\mathbf{Q}$.) This action on $\mathcal{A}_2(3)$ arises explicitly from the action of G on the 3-torsion $A[3] = (\mathbf{Z}/3\mathbf{Z})^2 \oplus (\mu_3)^2 \simeq (\mathbf{Z}/3\mathbf{Z})^4$ over E. We will apply Theorem 2.1 to the corresponding twist $\mathcal{A}_2^*(\overline{\rho})$. We then make crucial use of very explicit description of the cohomology of this compactification given by Hoffman and Weintraub ([HW01]). We recall some facts from that paper here now. The Picard group of $\mathcal{A}_2^*(3)$ over $\overline{\mathbf{Q}}$ is a free $\mathbf{Z}$-module of rank 61. It is generated by two natural sets of classes. The first is a 40-dimensional space explained by the 40 connected components of the boundary. The second is a 45-dimensional space explained by divisors coming from Humbert surfaces. These are also in one to one correspondence with the 45 nodes on the Burkhardt quartic. Together, these generate the Picard group of $\mathcal{A}_2^*(3)$ over $\overline{\mathbf{Q}}$, which is free of rank 61. Indeed, the Betti cohomology of $\mathcal{A}_2^*(3)$ over $\mathbf{Z}$ is free of degrees $1, 0, 61, 0, 61, 0, 1$ for $i = 0, \ldots 6$ by [HW01, Theorem 1.1]. Furthermore, all of these classes are trivial under the action of G_E. Let

$$\varrho : G_E \to \mathrm{PSp}_4(\mathbf{F}_3)$$

denote the projectivization of the representation $\overline{\rho}$ restricted to E. Note that the assumption on the similitude character implies that this restriction is valued in $\mathrm{Sp}_4(\mathbf{F}_3)$. The group $\mathrm{PSp}_4(\mathbf{F}_3)$ acts over E on $\mathcal{A}_2^*(3)$ via automorphisms, and $\mathcal{A}_2^*(\overline{\rho})$ is the twist of $\mathcal{A}_2^*(3)$ by ϱ. The group $\mathrm{Pic}_{\overline{\mathbf{Q}}}\mathcal{A}_2^*(\overline{\rho})$ as a G_E-module is obtained by considering $\mathrm{Pic}_{\overline{\mathbf{Q}}}\mathcal{A}_2^*(3)$ as a G module and then obtaining the Galois action via the map $\varrho : G_E \to G$. Thus it remains to closely examine $\mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3))$ as a G-representation over $\mathbf{Z}$. In fact, we can quickly prove a weaker version of Theorem 1.2 using this representation over $\mathbf{Q}$. The group G admits a unique conjugacy class G_{45} of subgroups of index 45, but two conjugacy classes of index 40; let G_{40} denote the (conjugacy class of) subgroup which fixes a point in the tautological action of $G = \mathrm{PSp}_4(\mathbf{F}_3)$ on $\mathbf{P}^3(\mathbf{F}_3)$. The following is an easy consequence of the calculations of [HW01] (and is also confirmed by our `magma` code).

Lemma 2.2. *As $\mathbf{Q}[G]$-modules, there is an isomorphism of virtual representations*

$$H^2(\mathcal{A}_2^*(3), \mathbf{Q}) \simeq \mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3)) \otimes \mathbf{Q} = \mathbf{Q}[G/G_{40}] + \mathbf{Q}[G/G_{45}] - [\chi_{24}],$$

where χ_{24} is the unique absolutely irreducible 24-dimensional representation of G.

Now, assuming that ϱ is *surjective*, we can prove that $\mathcal{A}_2^*(\overline{\rho})$ is not rational simply by proving that χ_{24} is not virtually equal to a sum of permutation representations, or equivalently, that $\chi_{24} \in R_{\mathbf{Q}}(G)$ does not lie in the image of the Burnside group generated by permutation representations. But one may compute (using `magma` or otherwise) that the Burnside cokernel of G has order 2 and is generated by χ_{24}. So this proves the weaker version of Theorem 1.2 where 6 is replaced by 2, although it is softer in that it only needs the $\mathbf{Q}[G]$-representation rather than the $\mathbf{Z}[G]$-representation. This argument also applies if one only assumes that the image of ϱ in $G = \mathrm{PSp}_4(\mathbf{F}_3)$ is H, as long as the restriction of χ_{24} to H is still non-trivial in the Burnside cokernel, which it is for precisely 7 of the 116 conjugacy classes of subgroups of G.

2.3. **Cohomological Obstructions.** We suppose from now on that the image of $\varrho : G_E \to G = \mathrm{PSp}_4(\mathbf{F}_3)$ is the group $H \subset G$ for some H. A second way to prove that a representation is not similar to the zero representation is to use cohomology. If M is a permutation representation of H, then the restriction of M to any subgroup P is also a permutation representation, and thus a direct sum of representations of the form $\mathbf{Z}[P/Q]$ for subgroups Q of P. But then (by Shapiro's Lemma) $H^1(P, M)$ is a direct sum of groups of the form

$$H^1(P, \mathbf{Z}[P/Q]) = H^1(Q, \mathbf{Z}) = 0,$$

where the second group vanishes because Q is finite. Moreover, the $\mathbf{Z}$ dual $M^\vee = \mathrm{Hom}(M, \mathbf{Z})$ of a permutation representation is isomorphic to the same permutation representation (a permutation matrix is its own inverse transpose). Thus one immediately has the following elementary criterion:

Lemma 2.4 (Cohomological Criterion for non-rationality)**.** *Suppose $\mathcal{A}_2^*(\overline{\rho})$ is rational over $E = \mathbf{Q}(\sqrt{-3})$. If $M = \mathrm{Pic}_{\overline{\mathbf{Q}}}(\mathcal{A}_2^*(3))$ as a G-module, and $\varrho|_{G_E}$ has image $H \subset G$, then*

$$H^1(P, M^\vee) = H^1(P, M) = 0$$

for every subgroup $P \subset H$.

We note that this is not an "if and only if" criterion. In the language of [CTS77], this is equivalent to saying that M as a G_E-representation is *flasque* and *coflasque* respectively, which (in general) is weaker than being stably permutation (which itself is not enough to formally imply rationality).

In order to test this criterion in practice, we need an explicit description of M as a $\mathbf{Z}[G]$-module rather than a $\mathbf{Q}[G]$-module. In order to do this, we explain how an explicit description of M can be extracted from Theorem 4.9 of [HW01]. That theorem describes a set of elements which generate both $H_4(\mathcal{A}_2^*(3), \mathbf{Z})$ and $H^2(\mathcal{A}_2^*(3), \mathbf{Z})$, and explicitly gives the intersection pairing between them. Moreover, the basis comes with a transparent action of the group G. Specifically, $H^2(\mathcal{A}_2^*(3), \mathbf{Z})$ is given as a quotient of $\mathbf{Z}[G/G_{40}] \oplus \mathbf{Z}[G/G_{45}]$. Hence to compute $H^2(\mathcal{A}_2^*(3), \mathbf{Z})$ as a G-module, it suffices to compute the quotient of $\mathbf{Z}[G/G_{40}] \oplus \mathbf{Z}[G/G_{45}]$ by the saturated subspace which pairs trivially with all elements of $H_4(\mathcal{A}_2^*(3), \mathbf{Z})$. Having carried out this computation, we obtain a (very) explicit free abelian group of rank 61 with an explicit action of G. We then do the following for every conjugacy class of subgroups $H \subset G$.

(1) Determine whether χ_{24} is non-trivial in the Burnside cokernel of H.
(2) Determine whether $H^1(P, M) \neq 0$ for any subgroup $P \subset H$.
(3) Determine whether $H^1(P, M^\vee) \neq 0$ for any subgroup $P \subset H$.

If any of these are non-trivial, this proves that $\mathcal{A}_2^*(\overline{\rho})$ is not rational. Moreover, a pullback–pushforward argument shows that the minimal degree of any rational covering is at least divisible by the exponent of $H^1(P, M)$ for any P.

We give one final statement which can be extracted from magma using the accompanying file but not directly from the table.

Lemma 2.5. *Suppose that the image of* $\overline{\rho}$ *contains an element conjugate in* $\mathrm{PSp}_4(\mathbf{F}_3)$ *to*

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{A}_2(\overline{\rho})$ *is not rational, and the minimal degree of any rational cover is divisible by* 3.

Proof. It suffices to note that this element generates the subgroup labelled as subgroup 6 in the table below, and then to apply Lemma 2.4. □

3. The Computation

Let $M = H^2(\mathcal{A}_2^*(3), \mathbf{Z})$. We have, by Poincaré duality, an isomorphism $M^\vee = H^4(\mathcal{A}_2^*(3), \mathbf{Z})$, Below we present in a table the result of our computation for all 116 subgroups of G, indicating the following data:

(1) The ordering $n = 1 \ldots 116$ of the conjugacy class of subgroup H as determined by `magma` with `SetSeed(1)`.
(2) The order of H.
(3) The group H in the small groups database [BEO01].
(4) The order in the Burnside cokernel over $\mathbf{Q}$ (if it is non-trivial).
(5) The lowest common multiple of the exponents of both $H^1(P, M)$ and $H^1(P, M^\vee)$ as P ranges over subgroups of $P \subset H$. If this is > 1, then the corresponding twist is not rational over E (or $\mathbf{Q}$). In particular, the fact that this number is 6 for G itself proves Theorem 1.2.
(6) The pre-image of H in $\mathrm{Sp}_4(\mathbf{F}_3)$ acts on $\mathbf{F}_3^4$. Is this action absolutely irreducible? (That is, is the action on $\overline{\mathbf{F}}_3^4$ irreducible.)
(7) A list of the conjugacy class of maximal subgroups of H (as indexed in the table). This allows one to compute the LCM column directly. The table is separated into blocks to reflect the geometry of the corresponding poset of subgroups. In particular, all maximal subgroups of H occur in blocks before that of H.
(8) The last two columns give $H^1(H, M)$ and $H^1(H, M^\vee)$.

The magma code accompanying this note computes G and M directly from the description given by Hoffman and Weintraub [HW01]. This leads to a representation of G as generated by two sparse 61×61 matrices x and y in $\mathrm{GL}_{61}(\mathbf{Z})$ such that the underlying module on which G acts (on the right, by magma conventions) is M. The matrices x and y are also printed in the output file of our magma script. The ordering of the subgroups is determined by the following `magma` code:

```
G<x,y>:=MatrixGroup<61,Integers()|x,y>;
SetSeed(1);
SL:=SubgroupLattice(G);
ZG:=Subgroups(G);
```

Note that other descriptions of G (even with the same seed) lead to other ordering of the subgroups. In fact, it's important to first compute `SubgroupLattice(G)` and then `Subgroups(G)`, since dropping the first will result in a different ordering.

n	$\lvert H \rvert$	SmallGroup	B	LCM	irred	maximal subgroups	$H^1(H,M)$	$H^1(H,M^\vee)$
1	1	<1,1>		1	no			
2	2	<2,1>		1	no	1		
3	2	<2,1>		1	no	1		
4	3	<3,1>		1	no	1		
5	3	<3,1>		1	no	1		
6	3	<3,1>		3	no	1	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$
7	5	<5,1>		1	no	1		
8	4	<4,1>		1	no	2 3		
9	4	<4,2>		1	no	2		
10	4	<4,2>		2	no	3		$(\mathbf{Z}/2\mathbf{Z})^2$
11	4	<4,2>		2	no	2 3		$\mathbf{Z}/2\mathbf{Z}$
12	4	<4,2>		1	no	3		
13	4	<4,1>		1	no	3		
14	6	<6,1>		3	no	2 6	$\mathbf{Z}/3\mathbf{Z}$	
15	6	<6,2>		1	no	2 4		
16	6	<6,2>		3	no	2 6		
17	6	<6,1>		3	no	3 6		$\mathbf{Z}/3\mathbf{Z}$
18	6	<6,1>		1	no	3 5		
19	6	<6,2>		1	no	2 5		
20	6	<6,2>		1	no	3 5		
21	9	<9,2>		3	no	5 6		$(\mathbf{Z}/3\mathbf{Z})^2$
22	9	<9,2>		3	no	4 6		$(\mathbf{Z}/3\mathbf{Z})^2$
23	9	<9,2>		3	no	4 5 6		
24	9	<9,1>		1	no	4		
25	10	<10,1>		1	no	3 7		
26	8	<8,4>		1	no	9		
27	8	<8,5>		2	no	11 12		$(\mathbf{Z}/2\mathbf{Z})^2$
28	8	<8,5>		2	no	10 11		$(\mathbf{Z}/2\mathbf{Z})^2$
29	8	<8,5>		2	no	8 10 11		
30	8	<8,2>		2	no	9 11		
31	8	<8,2>		2	no	11 13	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$
32	8	<8,3>		2	no	9 11		$\mathbf{Z}/2\mathbf{Z}$
33	8	<8,3>		2	no	10 12 13		$\mathbf{Z}/2\mathbf{Z}$
34	8	<8,3>		1	no	8 12 13		
35	12	<12,3>		2	no	5 10		
36	12	<12,3>		3	no	6 12	$\mathbf{Z}/3\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$
37	12	<12,4>		3	no	8 14 16 17		
38	12	<12,5>		1	no	8 19 20		
39	12	<12,1>		1	no	13 20		
40	12	<12,2>		1	no	9 15		
41	12	<12,4>		1	no	12 18 20		
42	18	<18,4>		3	no	17 18 21		$(\mathbf{Z}/3\mathbf{Z})^2$
43	18	<18,3>		3	no	14 16 21		
44	18	<18,3>		3	no	14 19 23		
45	18	<18,3>		3	no	14 15 22		
46	18	<18,3>		3	no	18 20 21		$\mathbf{Z}/3\mathbf{Z}$
47	18	<18,3>		3	no	17 20 23		

48	18	<18,5>		3	no	15 16 19 23		
49	20	<20,3>		1	yes	13 25		
50	27	<27,5>		3	no	21 22 23		$\mathbf{Z}/3\mathbf{Z}$
51	27	<27,3>		3	no	22		$(\mathbf{Z}/3\mathbf{Z})^2$
52	27	<27,4>		3	no	22 24		$\mathbf{Z}/3\mathbf{Z}$
53	16	<16,14>		2	yes	28 29		
54	16	<16,13>		2	no	26 30 32		
55	16	<16,11>		2	yes	27 28 30 32		$\mathbf{Z}/2\mathbf{Z}$
56	16	<16,3>		2	no	28 31		$(\mathbf{Z}/2\mathbf{Z})^2$
57	16	<16,11>		2	yes	27 29 31 33 34		$\mathbf{Z}/2\mathbf{Z}$
58	16	<16,3>		2	no	29 30 31		
59	24	<24,3>		1	no	15 26		
60	24	<24,13>		2	no	20 29 35		
61	24	<24,3>		3	no	16 26		
62	24	<24,3>		1	no	19 26		
63	24	<24,11>		1	no	26 40		
64	24	<24,13>		2	no	19 28 35		
65	24	<24,13>		6	no	16 27 36		
66	24	<24,12>		2	no	18 33 35		
67	24	<24,12>		6	no	17 33 36		$\mathbf{Z}/6\mathbf{Z}$
68	24	<24,12>		3	no	14 34 36	$\mathbf{Z}/3\mathbf{Z}$	
69	24	<24,8>		1	no	34 38 39 41		
70	36	<36,10>		3	no	37 42 43		
71	36	<36,10>		3	no	41 42 46		$\mathbf{Z}/3\mathbf{Z}$
72	36	<36,9>		3	no	13 42		$\mathbf{Z}/3\mathbf{Z}$
73	36	<36,12>		3	no	37 38 44 47 48		
74	54	<54,8>		3	no	45 51		
75	54	<54,13>		3	no	42 46 47 50		$\mathbf{Z}/3\mathbf{Z}$
76	54	<54,12>		3	no	43 44 45 48 50		
77	60	<60,5>		2	no	18 25 35		
78	60	<60,5>		3	no	17 25 36		$\mathbf{Z}/3\mathbf{Z}$
79	81	<81,7>		3	no	50 51 52		$\mathbf{Z}/3\mathbf{Z}$
80	32	<32,49>		2	no	54 55		
81	32	<32,6>		2	yes	55 56		$\mathbf{Z}/2\mathbf{Z}$
82	32	<32,27>		2	yes	53 55 56 57 58		
83	48	<48,30>		2	no	39 58 60		
84	48	<48,49>		2	yes	38 53 60 64		
85	48	<48,33>		2	yes	40 54 59		
86	48	<48,48>		2	no	41 57 60 66		$\mathbf{Z}/2\mathbf{Z}$
87	48	<48,48>		6	yes	37 57 65 67 68		
88	72	<72,40>		3	no	34 70 71 72		
89	72	<72,25>	2	3	no	48 59 61 62 63		
90	80	<80,49>		2	yes	7 53		
91	108	<108,40>		3	no	71 75		$\mathbf{Z}/3\mathbf{Z}$
92	108	<108,15>		3	no	40 74		
93	108	<108,38>		3	no	39 72 75		
94	108	<108,37>		3	no	70 73 75 76		
95	120	<120,34>		3	yes	37 49 68 78		

96	120	<120,34>		2	yes	41 49 66 77		
97	162	<162,10>		3	no	74 76 79		
98	64	<64,138>		2	yes	80 81 82		
99	96	<96,204>		2	no	62 64 80		
100	96	<96,204>		6	no	61 65 80		
101	96	<96,201>	2	2	no	63 80 85		
102	96	<96,195>		2	yes	69 82 83 84 86		
103	160	<160,234>		2	yes	25 82 90		
104	216	<216,88>	2	3	no	63 92		
105	216	<216,158>		3	no	69 88 91 93 94		
106	324	<324,160>		3	no	36 79 91		$\mathbf{Z}/3\mathbf{Z}$
107	360	<360,118>		6	no	66 67 72 77 78		$\mathbf{Z}/3\mathbf{Z}$
108	192	<192,1493>		6	yes	87 98 100		
109	192	<192,201>		2	yes	84 98 99		
110	288	<288,860>	2	6	no	89 99 100 101		
111	648	<648,553>	2	3	no	89 97 104		
112	648	<648,704>		3	no	68 97 105 106		
113	720	<720,763>		6	yes	86 87 88 95 96 107		
114	576	<576,8277>	2	6	yes	73 108 109 110		
115	960	<960,11358>		2	yes	77 102 103 109		
116	25920	G	2	6	yes	111 112 113 114 115		

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