Quadratic Chabauty for Atkin-Lehner Quotients of Modular Curves of Prime Level and Genus 4, 5, 6

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Abstract

We use the method of quadratic Chabauty on the quotients $X_0^+ (N)$ of modular curves $X_0 (N)$ by their Fricke involutions to provably compute all the rational points of these curves for prime levels $N$ of genus four, five, and six. We find that the only such curves with exceptional rational points are of levels 137 and 311. In particular there are no exceptional rational points on those curves of genus five and six. More precisely, we determine the rational points on the curves $X_0^+ (N)$ for $N = 137, 173, 199, 251, 311, 157, 181, 227, 263, 163, 197, 211, 223, 269, 271, 359$.

1 Introduction

The curve $X_0^+ (N)$ is a quotient of the modular curve $X_0 (N)$ by the Atkin-Lehner involution $w_N$ (also called the Fricke involution). The non-cuspidal points of $X_0^+ (N)$ classify unordered pairs of elliptic curves together with a cyclic isogeny of degree $N$ between them, where the Atkin-Lehner involution $w_N$ sends an isogeny to its dual. The set $X_0^+ (N) (\mathbb{Q})$ consists of cusps, points corresponding to CM elliptic curves (CM points), and points corresponding to quadratic $\mathbb{Q}$-curves without complex multiplication. The points of the third kind are referred to as exceptional points.

There have been several works related to the study of $\mathbb{Q}$-rational points on Atkin-Lehner quotients of modular curves (see [BDMTV21; Cas09; DL20; EL19; Gal96; Gal99; Gal02; Mer18; Mom87]) and on Atkin-Lehner quotients of Shimura curves (see [Cla03; PY07]). Especially relevant for this work are the articles [Gal96; Gal99; Gal02] in which Galbraith constructs models of all such curves of genus $\leq 5$ except for $X_0^+ (263)$, and conjectures that he has found all exceptional points on these curves. Building on work of Galbraith, Mercuri [Mer18] constructs models for such curves of genus 6 and 7 of prime level and shows that up to a (very) large naive height, there are no exceptional points on six of these curves (those with $N = 163, 197, 229, 269$ and 359).

Towards verifying Galbraith’s conjecture, Balakrishnan, Dogra, Müller, Tuitman, and Vonk [BDMTV21] show that of the curves $X_0^+ (N)$ of prime level $N$ and genus 2 and 3, the only curves with exceptional rational points are those with level $N = 73, 103, 191$. The rational points on $X_0^+ (N)$ for $N = \{67, 73, 103\}$ were computed by Balakrishnan, Best, Bianchi, Lawrence, Müller, Triantafillou, and Vonk [BBBLMTV19] using quadratic Chabauty and the Mordell-Weil

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sieve. The remaining genus 2 levels, \{107, 167, 191\}, are more challenging because there are not enough rational points to run the algorithm. Balakrishnan, Dogra, Müller, Tuitman, and Vonk [BDMTV21] compute the rational points on the remaining curves of genus 2 and prime level as well as those of genus 3 and prime level using models computed by Galbraith [Gal96] and Elkies.

Quadratic Chabauty refers to the technique of depth-two Chabauty-Kim. The Chabauty–Kim method [Kim05; Kim09] can be seen as a generalization of the usual Chabauty–Coleman method. The usual Chabauty–Coleman method is a useful tool in computing rational points on curves (see [Cha41; Col85; MP12; Sto19]), however there are still large Kim09] can be seen as a generalization of the usual Chabauty–Coleman method. The usual Chabauty–Coleman method developed in [BBBDLMT+20]. From Section 4, we have that for prime level \( N \) the curve \( X_0^+ (N) \) has genus 4 if and only if

\[
N \in \{137, 173, 199, 251, 311\}.
\]  

It has genus 5 if and only if

\[
N \in \{157, 181, 227, 263\},
\]

and it has genus 6 if and only if

\[
N \in \{163, 197, 211, 223, 269, 271, 359\}.
\]

These modular curves are not hyperelliptic according to [HH96]. They satisfy the conditions required for quadratic Chabauty, because their Mordell-Weil rank is equal to their genus and their Picard number is greater than 1. In addition, they usually have enough rational points (according to [BDMTV19, Remark 1.6], \( g + 1 \) points suffice when using equivariant \( p \)-adic heights, and one can lower this requirement by constructing \( g \) independent rational points on the Jacobian of the curve).

To use the quadratic Chabauty method, we use models for \( X_0^+ (N) \) computed by Galbraith [Gal96; Gal99; Gal02] and [Mer18]. Equations for \( N \in \{137, 157, 173, 181, 199, 227, 251\} \) are taken from [Gal96, pp. 32–34], the equation for \( N = 311 \) is taken from [Gal99, pp. 316], and equations for \( N \in \{163, 197, 211, 223, 269, 271, 359\} \) are taken from [Mer18, pp. 299–306]. The rational points are taken from [Gal96, pp. 88], [Gal99, pp. 316], and [Mer18, pp. 299–306]. For \( N \in \{199, 251\} \), we computed the known rational points by brute force. These equations and the list of known rational points in each of these levels are given in Section 5. With these models, we find suitable plane affine patches (in the sense of the last paragraph of Section 2) for the \textsc{QModAffine} code developed in [BBBDLMT+20] and verify that the list of rational points is complete.

Our main result is the following:

**Theorem 1.1.** For prime level \( N \), the only curves \( X_0^+ (N) \) of genus 4 that have exceptional rational points are \( X_0^+ (137) \) and \( X_0^+ (311) \). For prime level \( N \), there are no exceptional rational points on curves \( X_0^+ (N) \) of genus 5 and 6.

Therefore, this work confirms Galbraith’s conjecture for prime levels. In particular, this, combined with the results of [BDMTV21] on genus 2 and 3 curves, combined with works of [AM10; BBBLMTV19; BDMTV19; BGX19; Mom86; Mom87], and work in progress of the second author and Müller on curves \( X_0^+ (N) \) with composite level will resolve Galbraith’s conjecture. For genus 6, this work shows that the subset of rational points that Mercuri found is in fact all of them, and for those curves with \( N = 163, 197, 359 \) we confirm [Mer18, Result 1.2] without any restrictions on height.

This paper is organized as follows. In Section 2, we give an overview of the quadratic Chabauty method. Section 3 contains a description of the method we used, namely how we found the models that we used to run the quadratic Chabauty algorithm starting from the canonical model of \( X_0^+ (N) \) and how we chose suitable primes. In Section 4, we bound the genus of \( X_0^+ (N) \) from below and give a list of all levels \( N \) such that the genus of \( X_0^+ (N) \) is less than or equal to 6. In Section 5, we give tables for genus 4, 5, 6 containing the canonical models of \( X_0^+ (N) \), the known rational points, and our models and primes for which we run quadratic Chabauty.

\footnote{The canonical model for \( X_0^+ (157) \) given in Section 5 corrects a small typo in the third equation in Galbraith’s model for \( N = 157 \) (the leading term should be \( 2w^3 \) instead of \( w^3 \)).}
Acknowledgments

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2 Overview of Quadratic Chabauty for Modular Curves: theory and algorithm

Let $X/Q$ be a smooth projective geometrically connected curve of genus $g \geq 2$ with Jacobian $J$ whose Mordell-Weil group $J(Q)$ has rank $r = g$. For a prime $p$ of good reduction, the abelian logarithm induces a homomorphism

$$\log : J(Q_p) \to H^0(X_{Q_p}, \Omega^1)\vee.$$ 

We assume that the $p$-adic closure of the image of $J(Q) \subset J(Q_p)$ in $H^0(X_{Q_p}, \Omega^1)\vee$ has rank $g$ (otherwise classical Chabauty-Coleman applies). We also assume that $X(Q)$ is non-empty, so we can choose a rational base point $b$ to map $X$ into its Jacobian $J$. If the Néron-Severi rank $\rho$ of $J$ is larger than 1, then there exists a non-trivial $Z \in \text{Ker(NS}(J) \to \text{NS}(X))$ inducing a correspondence on $X \times_{Q} X$. Balakrishnan and Dogra in [BD18; BDMT19] explain how to attach to any such $Z$ a locally analytic quadratic Chabauty function

$$\rho_Z : X(Q_p) \to Q_p$$

as follows: using Nekovář’s theory of $p$-adic heights [Nek93], one can construct a global $p$-adic height which decomposes as a sum of local height functions. The quadratic Chabauty function $\rho_Z$ is defined as the difference between the global $p$-adic height and the local height for the chosen prime $p$. Even though we do not go into details here, we note that the computation of the global height pairing is easier when $X$ has many rational points (at least 3, which is indeed the case for all modular curves studied in this paper).

The crucial property of the quadratic Chabauty function is that there exists a finite set $Y \subset Q_p$, such that $\rho_Z(x) \in Y$ for any $x \in X(Q)$. Because $\rho_Z$ has Zariski-dense image on every residue disk and is given by a convergent power series, this implies that the set $X(Q_p)^2$ of $Q_p$-points of $X$ having values in $Y$ under $\rho_Z$ is finite. Because $X(Q)$ is contained in that set, it is finite as well. Since both $\rho_Z$ and $Y$ can be explicitly computed by [BBBDLMT+20], this makes the provable determination of $X(Q_p)^2$ possible\(^2\) if the quadratic Chabauty condition $r < g + \rho - 1$ is satisfied (e.g., if $r = g$ and $\rho > 1$).

In [BDMT21], the authors applied quadratic Chabauty to some of the modular curves associated to congruence subgroups of $\text{SL}_2(Z)$, as well as Atkin-Lehner quotients of such curves. The algorithm in [BDMT21] is specific to modular curves only when determining the nontrivial class $Z$. In their algorithm, $Z$ is computed by the Hecke operator $T_p$ (which is determined by the Eichler-Shimura relation). We recall the inputs and outputs of the algorithm in [BDMT21] here.

- **Input:**
  - A modular curve $X/Q$ that satisfies $r = g \geq 2$ and such that the $p$-adic closure of the image of $J(Q)$ in $H^0(X_{Q_p}, \Omega^1)\vee$ under log has rank $g$.
  - A prime $p$ which is a prime of good reduction for $X/Q$ such that the Hecke operator $T_p$ generates $\text{End}(J) \otimes Z$.
    We check this condition in our cases by showing that $a_p(f)$ generates the number field generated by the coefficients of $f$ where $f$ is any newform orbit representative associated to $X/Q$.

- **Output:** A finite set containing $X(Q_p)^2$.

The curves of our interest, $X^+_0(N)$ of genus 4, 5 and 6 and prime level $N$, satisfy the condition $r = g$. This is done by checking that for all $N$ in (1), (2) and (3), $f \in S_2(\Gamma_0(N))^{+\text{new}}$ satisfies $L'(f, 1) \neq 0$ (actually, we just need to check this for one arbitrary representative in each Galois orbit). According to [DL20, Proposition 7.1] (which the authors attribute to Kolyvagin and Logachev [KL89]), this implies that the (algebraic and analytic) rank of $J^+_0(N)/Q$ equals $\sum \dim(A_{f_j}) = g$ where the summation is taken over an arbitrary set of newform orbits representatives. However, it

\(^2\)at least under the assumption that $\rho_Z$ has no repeated roots; otherwise, we get a finite superset.
is not necessarily the case that \( r = g \) for all \( X_0^+(N) \) of prime level \( N \). The smallest genus \( g \) for which there exists a prime \( N \) such that \( r > g \) is \( g = 206 \), with \( N = 5077 \). [BDMTV21; DL20].

The implementation (available at [BBBDLMT+20]) of the algorithm in [BDMTV21] is designed to take as input a plane affine patch \( Y : Q(x, y) = 0 \) of a modular curve \( X/Q \). The model \( Q(x, y) = 0 \) does not need to be smooth, but it should be monic in \( y \) with \( p \)-integral coefficients. We can sometimes find an affine patch \( Y \) such that all rational points on \( X \) must be among the points returned by running their program on \( Y \). We have managed to do that for all genus 4 and 5 curves \( X_0^+(N) \) of prime level \( N \). If no such \( Y \) is found, then we find two affine patches such that every rational point on \( X \) is contained in at least one patch. Moreover, we need every \( \mathbb{F}_p \)-point on \( X \) (realized as a canonical model) to map to a good point (Definition 3.1) on at least one patch. Finding favorable affine patches turned out to be the most challenging part when applying their algorithm for large genus Atkin-Lehner quotients. We talk about our strategy for generating these affine patches in Section 3.

### 3 Overview of strategy

As introduced in Section 2, our first task is to find a suitable plane model \( Q(x, y) \) of \( X_0^+(N) \) that is monic in \( y \) and has “small” coefficients. We start with the image of the canonical embedding of \( X_0^+(N) \) in \( \mathbb{P}^{g-1} \) (as stated in the introduction, these are non-hyperelliptic for the levels \( N \) we consider). In what follows we shall identify \( X_0^+(N) \) with its canonical image and use the same notation for both. We find two rational maps \( \tau_x, \tau_y : X_0^+(N) \to \mathbb{P}^1 \) such that the product \( \tau_x \times \tau_y : X_0^+(N) \to \mathbb{P}^1 \times \mathbb{P}^1 \) is a birational map onto its image. Compose with the Segre embedding \( \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3_{[w:x:y:z]} \), and then project from the point \([1:0:0:0]\) onto the plane \( w = 0 \): denote by \( \varphi' \) the composite

\[
\varphi' : X_0^+(N) \xrightarrow{\tau_x \times \tau_y} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{Segre}} \mathbb{P}^3_{[w:x:y:z]} \xrightarrow{\text{projection}} \mathbb{P}^2_{[x:y:z]}.
\]

Choose \( x_1, x_2, y_1, y_2 \in \mathcal{O}_{\mathbb{P}^{g-1}}(1) \) such that \( \tau_x(q) = [x_1(q) : x_2(q)] \) and \( \tau_y(q) = [y_1(q) : y_2(q)] \) in coordinates. Then the equation for \( \varphi' \) is

\[
\varphi' : q \mapsto [(x_1y_2)(q) : (x_2y_1)(q) : (x_2y_2)(q)].
\]

When \( x_2(q), y_2(q) \neq 0 \), this is just \( \varphi' : q \mapsto [(x_1(x_2(q))/(y_1(y_2(q)) : 1).]

Note that on some closed subvarieties of \( X_0^+(N) \), the map \( \varphi' \) is not defined. Indeed, on the subvariety with additional equations \( x_1 = x_2 = 0 \), the map \( \tau_x \) is not defined; with \( y_1 = y_2 = 0 \), the map \( \tau_y \) is not defined; with \( x_2 = y_2 = 0 \), the map \( \tau_x \) or \( \tau_y \) or the projection following the Segre embedding is not defined. We denote the union of the three subvarieties by \( X_0^+(N) \varphi'_{\text{ndef}} \), outside which \( \varphi' \) is well-defined. The set of rational points \( X_0^+(N) \varphi'_{\text{ndef}}(\mathbb{Q}) \) can be computed directly using Magma [BCP97], and in our final step, we check that it does not contain any additional points other than the known rational points. For now, we focus on the generic part of \( X_0^+(N) \) where \( \varphi' \) is defined.

The image \( C'_N := \varphi'(X_0^+(N)) \) will be a curve given by an equation of the form

\[
Q_0(x, z)y^d + Q_1(x, z)y^{d-1} + \cdots + Q_d(x, z) = 0,
\]

where \( Q_0(x, 1) \) is monic. (If not, simply divide the equation throughout by the leading coefficient of \( Q_0(x, 1) \) to get to this form.) Multiply the above equation throughout by \( Q_0^{-1} \), we get

\[
(Q_0(x, z)y)^d + Q_1(x, z)(Q_0(x, z)y)^{d-1} + \cdots + Q_d(x, z)^{d-1}Q_d(x, z) = 0,
\]

Let \( \deg Q_0 \) denote the total degree of \( Q_0 \) and let \( C_N \) be the image of \( C'_N \) under the map

\[
\psi : [x : y : z] \mapsto [x^\deg Q_0 : Q_0(x, z)y : z^\deg Q_0 + 1].
\]

Then the affine patch of \( C_N \) given by \( z = 1 \) will have an equation \( Q(x, y) = 0 \) where \( Q(x, y) \) is a polynomial over \( \mathbb{Q} \) monic in \( y \), suitable for the quadratic Chabauty algorithm. Let \( \varphi : X_0^+(N) \to C_N \) denote the composite \( \varphi = \psi \circ \varphi' \).

Next, we select a suitable prime \( p \) and run QCModAffine on the affine patch of \( C_N \) given by \( z = 1 \). However, QCModAffine only computes rational points (and compares with known rational points) outside the bad residue disks.

**Definition 3.1** ([BT20], Definition 2.8 and 2.10). Let \( X_{\text{an}} \) denote the rigid analytic space over \( \mathbb{Q}_p \) associated to a projective curve \( X/Q_p \), given by (inhomogeneous) equation \( Q(x, y) = 0 \). Let \( \Delta(x) \) be the discriminant of \( Q \) as a function of \( y \), and let \( r(x) = \Delta/\gcd(\Delta, d\Delta/dx) \). We say that a residue disk (as well as any point inside it) is infinite if it contains a point whose \( z \)-coordinate is \( \infty \), is bad if it contains a point whose \( z \)-coordinate is \( \infty \) or a zero of \( r(x) \), and is good if it is not bad.
Therefore, we may need to repeat this construction multiple times (with the same $p$), i.e., we need a collection $$\{(\varphi_{N,i}, C_{N,i})\}_{i=1}^k$$ such that for each $P \in X_0^+(N)(F_p)$, there exists some $i$ such that $\varphi_{N,i}(P)$ does not map to a bad $F_p$-point. This would imply that for each residue disk $D$ of $X_0^+(N)(Q_p)$, there exists some $i$ such that $\varphi_{N,i}(D) \subseteq C_{N,i}(Q_p)$ is contained in a good residue disk. If $\text{QModAffine}$ reports that the only rational points in the good disks of $C_{N,i}(Q_p)$ are the images of the known rational points, then we know that $X_0^+(N)(Q)$ is contained in the finite set $$\bigcup_{i=1}^k \varphi_{N,i}^{-1}(\varphi_{N,i}(X_0^+(N)(Q)_{\text{known}})) \cup X_0^+(N,R_{\text{underf}}(Q),$$

and then we check that this equals $X_0^+(N)(Q)_{\text{known}}$ using direct Magma computations.

In practice, we seek to choose $x_1, x_2, y_1, y_2 \in \mathcal{O}_{p_s-1}(1)$ so that $d_x := \deg r_x, d_y := \deg r_y$ are small for faster computations, since the degree of the defining equation $Q(x, y)$ of $C_N$ is at most $d_x d_y$ and the maximum power of $y$ that appears is $d_x$. We would also like to keep $k$ small, and ideally to have $k = 1$, in which case we need to find a prime $p$ for which $Q(x, y) = 0$ has no bad disks. In order to do so, we would like to start with $Q(x, y)$ for which $r(x)$ has no linear factor over $Q$. If the construction does not satisfy this property, we adjust the construction of $\varphi'$ by applying some $\sigma \in \text{Aut}(\mathbb{P}^1 \times Q \mathbb{P}^1)$ before the Segre embedding, and/or by post-composing $\varphi'$ by some $\rho \in \text{Aut}(\mathbb{P}^2)$ (the latter only invoked in the $N = 211$ case). With suitable adjustments in the construction of $\varphi'$, we were able to find plane models $Q(x, y)$ for which $r(x)$ has no linear factor over $Q$ for all the cases we aim to solve in this paper. Even so, for four of the genus 6 cases, $N = 197, 211, 223, 359$, we did not find a model for which there is a prime $p$ without any bad disks, and in each of these four cases, we were able to find two patches and a prime $p$ such that for each $P \in X_0^+(N)(F_p)$, $\varphi_{N,i}(P)$ does not map to a bad $F_p$-point on at least one of the patches. For all the other prime levels where $X_0^+(N)$ has genus 4, 5, 6, we were able to find a single patch and a prime for which there is no bad disk.

That is, we were able to solve the problem with $k \leq 2$.

In this section, we do not use that the curve in question has a modular interpretation. The strategy could possibly be applied to other curves of arbitrary genus $g$. Currently Magma supports computing plane models and gonal maps for curves up to genus 6.

## 4 Modular curves $X_0^+(N)$ of genus at most 6

We will obtain an explicit lower bound on the genus of $X_0^+(N)$ in order to find all $X_0^+(N)$ of genus at most 6.

Let $g_0(N)$ be the genus of $X_0(N)$ and let $g_0^+(N)$ be the genus of $X_0^+(N)$.

**Theorem 4.1.** For integers $N \geq 1$, we have $g_0(N) \geq (N - 5\sqrt{N} - 8)/12$.

*Proof.* See Section 3 of [CWZ00].

Let $\nu(N)$ denote the number of fixed points of $w_N$. Let $h(D)$ denote the class number of the quadratic order with discriminant $D$.

**Theorem 4.2.** For $N \geq 5$,

$$\nu(N) = \begin{cases} h(-4N) + h(-N) & \text{if } N \equiv 3 \pmod{4} \\ h(-4N) & \text{otherwise} \end{cases}$$

*Proof.* Apply the formula in Remark 2 of [FH99].

To bound $g_0^+(N)$, we will need an upper bound on $h(D)$.

**Lemma 4.3.** For negative $D$,

$$h(D) \leq \frac{\sqrt{-D}}{\pi} (\ln(4|D|) + 2).$$

*Proof.* For negative $D$, the Dirichlet class number formula states that

$$h(D) = \frac{w\sqrt{-D}}{2\pi} L(1, \chi_D) \quad (4)$$

where $\chi_D(m) = \left( \frac{D}{m} \right)$ is the Kronecker symbol and

$$w = \begin{cases} 4 & \text{if } D = -4, \\ 6 & \text{if } D = -3, \\ 2 & \text{otherwise.} \end{cases}$$
Hence, it suffices to obtain an upper bound on \( L(1, \chi_D) \). Let \( A(x) := \sum_{1 \leq n \leq x} \chi_D(n) \). Since the modulus of \( \chi_D \) divides \( 4D \), the sum of any \( 4|D| \) consecutive values of \( \chi_D(n) \) is zero, so

\[
|A(x)| \leq 4|D|.
\]

Using the definition of \( L(1, \chi_D) \) and the Abel summation formula,

\[
L(1, \chi_D) = \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n} = \sum_{n=1}^{4|D|} \frac{\chi_D(n)}{n} + \sum_{n>4|D|} \frac{\chi_D(n)}{n}
\]

\[
= \frac{4|D|}{n} - \frac{A(4D)}{4D} - \int_{4|D|}^{\infty} \frac{A(t)}{t^2} dt.
\]

(Note that the series for \( L(1, \chi_D) \) converges because \( \chi_D \) is nontrivial.) Since \( A(4D) = 0 \), the middle term vanishes. Using the upper bounds \( |\chi_D(n)| \leq 1 \) and \( |A(x)| \leq 4|D| \) yields

\[
L(1, \chi_D) \leq \frac{4|D|}{n} + 4|D| \int_{4|D|}^{\infty} \frac{1}{t^2} dt
\]

\[
= \sum_{n=1}^{4|D|} \frac{1}{n} + 1
\]

\[
\leq \ln(4|D|) + 2
\]

from the Abel summation formula again. Substituting into the Dirichlet class number formula (4) shows that for \( D \not\in \{-3, -4\} \),

\[
h(D) \leq \frac{\sqrt{-D}}{\pi} (\ln(4|D|) + 2).
\]

We then manually check that this holds for \( D \in \{-3, -4\} \) as well.

\[\square\]

**Proposition 4.4.** We have

\[
g_0^+(N) \geq \frac{N - 5\sqrt{N} + 4}{24} - \frac{\sqrt{N}}{\pi} (\ln(16N) + 2).
\]

**Proof.** From the Riemann-Hurwitz formula applied to the degree-2 morphism \( X_0(N) \to X_0^+(N) \), we have

\[
2g_0(N) - 2 = 2(2g_0^+(N) - 2) + \nu(N),
\]

and we are done by combining Theorem 4.1, Theorem 4.2, and Lemma 4.3.

\[\square\]

**Proposition 4.5.** The complete list of levels \( N \) for which \( g_0^+(N) \leq 6 \) is as follows.

<table>
<thead>
<tr>
<th>( g_0^+(N) )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71</td>
</tr>
<tr>
<td>1</td>
<td>37, 43, 53, 61, 79, 83, 89, 101, 131</td>
</tr>
<tr>
<td>2</td>
<td>67, 73, 103, 107, 167, 191</td>
</tr>
<tr>
<td>3</td>
<td>97, 109, 113, 127, 139, 149, 151, 179, 239</td>
</tr>
<tr>
<td>4</td>
<td>137, 173, 199, 251, 311</td>
</tr>
<tr>
<td>5</td>
<td>157, 181, 227, 263</td>
</tr>
<tr>
<td>6</td>
<td>163, 197, 211, 223, 269, 271, 359</td>
</tr>
</tbody>
</table>

Table 1: The prime levels \( N \) such that \( X_0^+(N) \) has genus \( \leq 6 \)
In the following tables, we display our choice of models for \( g_0(N) \) respectively. 

\[
\begin{array}{c|ccccccc}
N & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
p & 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 26, 27, 32, 35, 36, 39, 49, 50 \\
p & 22, 28, 30, 33, 34, 38, 40, 44, 45, 48, 51, 54, 55, 56, 63, 64, 65, 75, 81, 95, 119 \\
p & 42, 46, 52, 57, 62, 68, 69, 72, 74, 77, 80, 87, 91, 98, 111, 121, 125, 143 \\
p & 58, 60, 66, 76, 85, 86, 96, 99, 100, 104, 128, 169 \\
p & 70, 82, 84, 88, 90, 92, 93, 94, 108, 115, 116, 117, 129, 135, 147, 155, 159, 161, 215 \\
p & 78, 105, 106, 110, 112, 122, 123, 133, 134, 144, 145, 146, 171, 175, 185, 209 \\
p & 118, 124, 136, 141, 152, 153, 164, 183, 203, 221, 299 \\
\hline
\end{array}
\]

Table 2: The composite levels \( N \) such that \( X_0^+(N) \) has genus \( \leq 6 \)

\[\text{Proof.} \] The lower bound in Proposition 4.4 exceeds 6 for \( N > 13300 \). For \( N \leq 13300 \), we compute \( g_0(N) \) exactly from the computation of \( g_0(N) \) in [DS05, §3.1], Theorem 4.2 (the class numbers of imaginary quadratic fields can be computed efficiently using Magma, e.g. via the identification with the number of equivalence classes of reduced integral binary quadratic forms) and (5) and find the \( N \) for which \( g_0(N) \leq 6 \).

5 Results and data

In the following tables, we display our choice of models for \( X_0^+(N) \) for which we applied QCModAffine successfully. When the rational point is a CM point, we use \( D \) to represent its discriminant. To compute the coordinates of a CM point of discriminant \( D \), we first compute the Heegner forms of elliptic curves with CM by an order of discriminant \( D \), and then the corresponding \( \tau \) in the upper half plane. Then, we substitute \( q = \exp(2\pi i \tau) \) into the \( q \)-expansions of the cusp forms giving the canonical embedding, to obtain approximations of the coordinates of the CM point in \( \mathbb{P}^{g-1} \). For certain values of \( D \) or \( N \), the series converges very slowly, but if there is only one such \( D \) for a given \( N \), and all \( \mathbb{Q} \)-points on \( X_0^+(N) \) are known, since we know the modular interpretation for all but one of them, we can also determine the interpretation of the remaining one. If all cusp forms vanish at the CM point corresponding to \( D \) (this happens for \( D = -3, -4 \) in the case of \( X_0^+(157) \) and \( X_0^+(181) \), and \( D = -3, -163 \) in the case of \( X_0^+(163) \)), we compute the derivatives of \( q \)-expansions of the basis of the cusp forms giving the canonical embedding. In this way, we get the coordinates of the CM points with \( D = -4 \) and \(-163 \).

Use \( d_\infty \) to denote the degree of the residue field at infinity. In the first table, we list \( p, d_\infty, d_x, d_y \) and the runtime for all affine patches used. The computations were performed on a six core server with 64 GB of RAM and processor Intel(R) Xeon(R) W-2133 CPU @ 3.60 GHz.

\[
\begin{array}{c|c|cccc|c}
N & p & d_\infty & d_x & d_y & \text{Runtime} \\
\hline
137 & 5 & 6 & 3 & 3 & 00m 36s 370ms \\
173 & 5 & 2 & 4 & 4 & 01m 58s 450ms \\
199 & 7 & 4 & 4 & 4 & 02m 23s 060ms \\
251 & 11 & 4 & 4 & 4 & 04m 39s 420ms \\
311 & 5 & 2 & 4 & 4 & 01m 53s 110ms \\
\hline
157 & 5 & 4 & 4 & 5 & 06m 12s 150ms \\
181 & 7 & 6 & 3 & 5 & 03m 50s 490ms \\
227 & 23 & 6 & 3 & 5 & 03m 19s 710ms \\
263 & 23 & 4 & 6 & 6 & 22m 47s 600ms \\
\hline
163 & 31 & 4 & 4 & 4 & 15m 59s 450ms \\
269 & 29 & 4 & 4 & 6 & 41m 23s 000ms \\
271 & 13 & 12 & 5 & 9 & 1235m 42s 060ms \\
197_1 & 23 & 2 & 6 & 6 & 24m 09s 610ms \\
197_2 & 23 & 8 & 5 & 6 & 18m 48s 930ms \\
211_1 & 31 & 4 & 6 & 6 & 58m 42s 680ms \\
211_2 & 31 & 2 & 6 & 6 & 34m 52s 620ms \\
223_1 & 19 & 12 & 6 & 5 & 26m 04s 680ms \\
223_2 & 19 & 2 & 6 & 5 & 18m 56s 530ms \\
359_1 & 7 & 4 & 6 & 6 & 9m 54s 270ms \\
359_2 & 7 & 4 & 6 & 5 & 15m 41s 880ms \\
\end{array}
\]

For \( N = 197, 211, 223, 359 \), runtimes are given for the first and the second patch listed in the tables later in this section, respectively.
The models we have used are now listed and organized by the genus. We also write down the classification of all rational points.

### 5.1 Genus 4

<table>
<thead>
<tr>
<th>$g = 4$, equations for canonical embedding in $\mathbb{P}^{8-1}$ are</th>
</tr>
</thead>
<tbody>
<tr>
<td>$XY + WY + 2Y^2 + 2ZW + XZ + 6YZ + 3Z^2 = 0,$</td>
</tr>
<tr>
<td>$X^3 + WX^2 + 6X^2Z - 2XY^2 - 5XYZ + XZW + 13XZ^2 + 2Y^3$</td>
</tr>
<tr>
<td>$+ 3WY^2 + W^2Y + 3WYZ - 6YZ^2 + ZW^2 - 4Z^2W + 14Z^3 = 0,$</td>
</tr>
<tr>
<td>with the variables corresponding to the following cusp forms</td>
</tr>
<tr>
<td>$W = -q + q^2 - q^3 + q^4 + 3q^5 + 2q^6 + 4q^7 - 3q^8 + 2q^9 - 5q^{10} - q^{11} - 2q^{12} + 4q^{13} - 5q^{14}$</td>
</tr>
<tr>
<td>$+ 3q^{15} + 2q^{16} - 3q^{17} - 2q^{18} - q^{19} + O(q^{20}),$</td>
</tr>
<tr>
<td>$X = -q^2 + 2q^3 + 2q^4 - 3q^5 - 2q^6 - 3q^7 + q^8 + q^9 + 4q^{10} + 10q^{11} - 2q^{12} - q^{13} + 6q^{14}$</td>
</tr>
<tr>
<td>$- 4q^{15} - 7q^{16} - 5q^{17} + q^{18} - 7q^{19} + O(q^{20}),$</td>
</tr>
<tr>
<td>$Y = 2q^3 - q^4 - 3q^5 - q^6 - q^7 + 3q^8 - 2q^9 + 6q^{10} + 3q^{11} + 3q^{12} - q^{13} + q^{14} - 3q^{16}$</td>
</tr>
<tr>
<td>$+ 2q^{17} + 3q^{18} + q^{19} + O(q^{20}),$</td>
</tr>
<tr>
<td>$Z = -q^3 + 2q^5 + 2q^6 + q^7 - q^8 - 3q^{10} - 4q^{11} - q^{12} - q^{14} + 3q^{16} + q^{17} - 2q^{18} + 2q^{19} + O(q^{20})$</td>
</tr>
<tr>
<td>9 known rational points are</td>
</tr>
</tbody>
</table>

For quadratic Chabauty, we use $p = 5$ and a single model $(\varphi_{137,1}, C_{137,1})$

Information for $\varphi_{137,1}$:

- $d_x = 3$
- $d_y = 3$
- $x_1 = Z$
- $x_2 = Y$
- $y_1 = (2 \cdot 3 \cdot 7)Z$
- $y_2 = W + X + 2Y + Z$

Equation of $C_{137,1}$ in the affine patch where $z = 1$ is

\[
y^3 + (50x^3 + 32x^2 - 4x - 3)y^2 + (966x^6 + 1377x^5 + 459x^4 - 115x^3 - 66x^2 + x + 2)y + (7056x^9 + 16128x^8 + 12744x^7 + 2856x^6 - 1239x^5 - 678x^4 - 35x^3 + 28x^2 + 4x) = 0.
\]

According to Galbraith [Gal99], the $j$-invariant of the $\mathbb{Q}$-curve corresponding to the exceptional point is

\[
j = (-423554849102365349285527612080396097711989843 + 9281040308790916967443095886224534005155665\sqrt{-31159})/2^{138}.
\]
with the variables corresponding to the following cusp forms

\[ W = q - 2q^4 - 2q^6 - 3q^7 - 4q^9 + 2q^{11} + q^{12} + 3q^{13} - q^{14} + q^{15} + 3q^{16} - 3q^{17} - q^{18} - 4q^{19} + O(q^{20}), \]

\[ X = -q^2 + 3q^7 + 4q^8 - q^9 + 2q^{10} - 6q^{11} - q^{12} - 10q^{13} + 2q^{14} - q^{15} - q^{16} + 7q^{17} + 3q^{18} + 2q^{19} + O(q^{20}), \]

\[ Y = -q^3 + q^4 + q^5 - q^6 + q^7 + 4q^9 - 4q^{11} - q^{12} - 5q^{13} - q^{15} - 3q^{16} + 5q^{17} + 4q^{18} + 3q^{19} + O(q^{20}), \]

\[ Z = q^6 - q^7 - q^8 - q^9 - q^{10} + 3q^{11} + q^{13} + q^{16} - 4q^{17} - 3q^{18} - 2q^{19} + O(q^{20}) \]

6 known rational points are


For quadratic Chabauty, we use \( p = 5 \) and a single model \( (\varphi_{173,1}, C_{173,1}) \)

Information for \( \varphi_{173,1} \): 

\[ d_x = 4 \quad d_y = 4 \]

\[ x_1 = W + X + 2Y + 4Z \]

\[ y_1 = (3 \cdot 7/2) \cdot (X - 3Y - 3Z) \]

\[ x_2 = W + Y \]

\[ y_2 = 2W + X + 9Y + 15Z \]

Equation of \( C_{173,1} \) in the affine patch where \( z = 1 \) is

\[ y^4 + \frac{1}{2^5 \cdot 3^2} (198x^4 - 51x^3 + 8x^2 - 495x + 54)y^3 \]

\[ + \frac{1}{2^8 \cdot 3^3} \left( \begin{array}{c}
133264x^8 - 15804x^7 + 1164x^6 - 166127x^5 + 81837x^4 \\
-18480x^3 + 180184x^2 - 44117x + 1743
\end{array} \right) y^2 \]

\[ + \frac{1}{2^3 \cdot 3^4} \left( \begin{array}{c}
2667168x^{12} - 1333584x^{11} - 1775088x^{10} - 17635032x^9 \\
+14542866x^8 - 2087801x^7 + 40446120x^6 - 27642183x^5 \\
+6246796x^4 - 27504295x^3 + 11701160x^2 - 957873x + 22050
\end{array} \right) y \]

\[ + \frac{1}{2^{12} \cdot 3^4} \left( \begin{array}{c}
192036096x^{16} - 132596352x^{15} - 350759808x^{14} - 1097128800x^{13} \\
+1384863264x^{12} + 522223848x^{11} + 3767950920x^{10} - 4672257034x^9 \\
+908642397x^8 - 5781010796x^7 + 5563290280x^6 - 1437899280x^5 \\
+291004918x^4 - 2024404284x^3 + 258937560x^2 - 12131910x + 194481
\end{array} \right) = 0. \]
$g = 4$, equations for canonical embedding in $\mathbf{P}^{3}$ are

\[
2WY - X^2 + 3XY - 6XZ - 5WZ + 3YZ - 6Z^2 = 0,
- WX^2 + 3WY^2 + 3X^2Y + 3WXY + 3Y^2 + Y^3 - 7WZ^2 - 5WXZ
- 8WYZ - 2XYZ - 3Y^2Z + 6WZ^2 + 4YZ^2 - 3Z^3 = 0,
\]

with the variables corresponding to the following cusp forms

\[
W = q + q^3 - 2q^4 - 2q^5 - q^6 - 6q^7 - 3q^9 + q^{10} + 2q^{11} - q^{12} + 3q^{13} + 2q^{14} + 5q^{16}
- q^{17} + q^{18} - 3q^{19} + O(q^{20}),
\]

\[
X = q^2 + q^3 - 3q^6 - 2q^7 - 4q^8 - q^9 + 2q^{10} + 3q^{11} + q^{12} + 5q^{13} + 2q^{14} + 3q^{15} + 4q^{16}
+ 4q^{17} + q^{18} - 4q^{19} + O(q^{20}),
\]

\[
Y = -2q^3 + q^4 + q^5 + q^6 + 5q^7 + 2q^9 - 3q^{10} - 4q^{11} + q^{12} - 3q^{13} - 2q^{14} + q^{15} - 6q^{16}
- 3q^{17} - q^{18} + 5q^{19} + O(q^{20}),
\]

\[
Z = -q^3 + q^5 + q^6 + 2q^7 + q^8 + q^9 - 2q^{10} - 2q^{11} - 2q^{13} - q^{14} - 3q^{16} - 2q^{17} - q^{18} + 3q^{19} + O(q^{20})
\]

8 known rational points are

\[
\begin{align*}
\text{Cusp, } [1: 0: 0: 0] & \quad D = -19, [1: -1: -1: 0] \\
D = -12, [1: 3: -1: -1] & \quad D = -163, [5: -2: -3: -1]
\end{align*}
\]

For quadratic Chabauty, we use $p = 7$ and a single model $(\varphi_{199,1}, C_{199,1})$

\begin{align*}
\text{Information for } \varphi_{199,1}: \\
d_x = 4 & \quad d_y = 4 \\
x_1 = W + Y & \quad y_1 = (5 \cdot 11) \cdot (Y - Z) \\
x_2 = W - X + 2Y - 3Z & \quad y_2 = X - 4Y + 7Z
\end{align*}

Equation of $C_{199,1}$ in the affine patch where $z = 1$ is

\[
y^4 + \left(188x^4 - 531x^3 + 730x^2 - 585x + 195\right)y^3
+ \left(6930x^8 - 36467x^7 + 102801x^6 - 201149x^5
+ 280108x^4 - 273201x^3 + 179227x^2 - 70731x + 12474\right)y^2
+ \left(87725x^{12} - 678755x^{11} + 2812416x^{10} - 8175060x^9
+ 17971418x^8 - 30862293x^7 + 42039858x^6 - 45235151x^5
+ 37608990x^4 - 23234260x^3 + 9991212x^2 - 2649537x + 323433\right)y
+ \left(332750x^{16} - 3454550x^{15} + 18928470x^{14} - 72428151x^{13}
+ 212751798x^{12} - 501847128x^{11} + 975020037x^{10} - 1580007615x^9
+ 2144054322x^8 - 2430428083x^7 + 2279776105x^6 - 1737806496x^5
+ 1046077245x^4 - 476258451x^3 + 153252000x^2 - 30873150x + 2910897\right) = 0.
\]
\( g = 4 \), equations for canonical embedding in \( \mathbb{P}^9 \) are

\[
WX - WY + XY - 2Y^2 + 2WZ + XZ + 4YZ + Z^2 = 0,
\]
\[
W^2X - W^2Y + X^2Y - WY^2 - 2XY^2 + 2Y^3 + W^2Z - WXZ
- 3X^2Z + 2WYZ + 7XYZ - 3Y^2Z - 21XZ^2 + 10YZ^2 - 28Z^3 = 0,
\]

with the variables corresponding to the following cusp forms

\[
W = -q - q^2 + 2q^4 + 3q^5 + q^7 + 3q^8 + 3q^9 + 2q^{10} - q^{11} + q^{13} + q^{14} + q^{15} - 3q^{16} - 6q^{17} + 4q^{19} + O(q^{20}),
\]
\[
X = q^2 - q^3 - q^4 + 2q^5 - q^6 - q^7 - 2q^8 + q^{10} - q^{11} + 2q^{12} - q^{14} + 3q^{16} - 6q^{17} + 4q^{19} + O(q^{20}),
\]
\[
Y = q^2 - q^4 + 2q^5 + q^6 + q^7 - 2q^8 - q^9 + 2q^{10} + 2q^{11} - q^{12} + q^{13} - 2q^{14} + 3q^{16} + 4q^{17} - 4q^{18} - 4q^{19} + O(q^{20}),
\]
\[
Z = -q^5 + q^6 + q^7 - q^{10} + q^{11} - q^{12} - q^{14} + 3q^{17} - q^{18} - 3q^{19} + O(q^{20})
\]

6 known rational points are

\[
X_0^+ (251)
\]

For quadratic Chabauty, we use \( p = 11 \) and a single model \((\varphi_{251,1}, C_{251,1})\)

Information for \(\varphi_{251,1}:\)

\[
d_x = 4 \quad \quad \quad d_y = 4
\]
\[
x_1 = W + 7X + 5Z \quad \quad \quad y_1 = (5/3)Z
\]
\[
x_2 = W - 6X + 10Y - 23Z \quad \quad \quad y_2 = X - Z
\]

Equation of \( C_{251,1} \) in the affine patch where \( z = 1 \) is

\[
y^4 + \frac{1}{2^2 \cdot 3 \cdot 5} \left( 235x^4 - 1400x^3 - 3188x^2 - 1402x - 330 \right) y^3
\]
\[
+ \frac{1}{2^4 \cdot 3^2 \cdot 5^2} \left( 16700x^8 - 247290x^7 + 176078x^6 + 2948981x^5 + 4658866x^4 + 3074269x^3 + 1328589x^2 + 339250x + 42800 \right) y^2
\]
\[
+ \frac{1}{2^6 \cdot 3^3 \cdot 5^3} \left( 5100000x^{12} - 12344000x^{11} + 61126700x^{10} + 217315540x^9 - 425974544x^8 - 2004263968x^7 - 2611092185x^6 - 1852500895x^5 - 1038555014x^4 - 475857025x^3 - 154375600x^2 - 29480000x - 2560000 \right) y
\]
\[
+ \frac{1}{2^8 \cdot 3^4 \cdot 5^4} \left( 60000000x^{16} - 203600000x^{15} + 1956240000x^{14} - 664896000x^{13} - 4134795500x^{12} - 66346728360x^{11} + 5942868304x^{10} + 181499926316x^9 + 43955025678x^8 - 137005175370x^7 - 105387676043x^6 + 907612425x^5 + 35725442325x^4 + 20724635625x^3 + 5829465000x^2 + 867000000x + 56000000 \right) = 0.
\]

11
According to Galbraith [Gal99], the $j$-invariant of the $\mathbb{Q}$-curve corresponding to the exceptional point is
\[ j = 31244183594433270730909857930589729152601677824000000 \\
\pm 1565810538998051715397339689492195035077551267840000\sqrt{39816211853}. \]
5.2 Genus 5

\[ g = 5, \text{ equations for canonical embedding in } \mathbb{P}^{7-1} \text{ are} \]

\[
VX - VY + WY - XY +YZ = 0, \\
V^2 - VW + WX + WY + XY + VZ - WZ + XZ = 0, \\
2VX + WX - VY - XY - VZ - WZ + 2XZ - Z^2 = 0,
\]

with the variables corresponding to the following cusp forms

\[
V = -q^4 + q^5 + q^6 + 2q^8 - q^9 - 2q^{10} - 2q^{11} - q^{13} + 2q^{14} - q^{15} - q^{17} - q^{18} + 3q^{19} + O(q^{20}) \\
W = -q^2 + q^3 + q^4 + q^5 + q^7 + 2q^8 - q^{10} + 2q^{11} + q^{13} + q^{15} - q^{16} + 3q^{17} - q^{18} + q^{19} + O(q^{20}) \\
X = -q^2 + q^3 + q^4 + q^5 + q^6 - 2q^9 + 2q^{10} + q^{12} - q^{13} + 2q^{14} - 2q^{15} - q^{16} - 4q^{17} + 2q^{18} + q^{19} + O(q^{20}) \\
Y = q^5 - q^6 - 2q^7 + q^8 + q^{11} + 2q^{12} + q^{13} + q^{14} - q^{15} - 3q^{16} - 2q^{17} + 3q^{18} - q^{19} + O(q^{20}) \\
Z = -q^2 + 2q^4 + q^5 + q^6 - q^7 - q^8 + q^9 + q^{10} + q^{11} - 2q^{12} - 2q^{15} - 2q^{17} - q^{18} - 2q^{19} + O(q^{20})
\]

9 known rational points are

Cusp, \([0:1:0:0:0]\) \quad \text{\(D = -16, [1:1:0:0:0]\)}

\(D = -3, [3:2:0:0:0-2:1]\)

\(D = -4, [1:1:0:0:0-2]\)

\(D = -11, [0:0:0:1:0]\)

\(D = -12, [1:0:0:0:0-1]\)

For quadratic Chabauty, we use \(p = 5\) and a single model \((\varphi_{157,1}, C_{157,1})\)

Information for \(\varphi_{157,1}\):

\[\begin{align*}
d_x & = 4 & d_y & = 5 \\
x_1 & = 3(V + 2Y + 4Z) & y_1 & = V - W + Y \\
x_2 & = V - 4W - 3X - 2Y - 5Z & y_2 & = 2V - 2W - Y
\end{align*}\]

Equation of \(C_{157,1}\) in the affine patch where \(z = 1\) is

\[\begin{align*}
y^4 + \frac{1}{2\cdot 3^2} (-52x^5 - 683x^4 - 3544x^3 - 9531x^2 - 12978x - 6885) y^3 \\
+ \frac{1}{2^2 \cdot 3^3} \left( 234x^{10} + 6461x^9 + 80107x^8 + 589432x^7 + 2842712x^6 + 9365724x^5 \right) y^2 \\
+ \frac{1}{2^3 \cdot 3^6} \left( 972x^{15} + 8640x^{14} - 474801x^{13} - 1305897x^{12} - 16243762x^{11} - 1259359425x^{10} \\
- 6767639914x^9 - 26534799963x^8 - 7795365944x^7 - 173540777940x^6 - 29246484754x^5 \\
- 368070954012x^4 - 335685400200x^3 - 209644813980x^2 - 80154057384x - 14129953308 \right) y \\
+ \frac{1}{2^4 \cdot 3^8} \left( -40824x^{20} - 1958580x^{19} - 42950790x^{18} - 571046195x^{17} - 513734177x^{16} \\
- 32912831723x^{15} - 152493625145x^{14} - 49644049072x^{13} - 96538029930x^{12} \\
+ 126852890994x^{11} + 9026198316090x^{10} + 3898387449316x^9 \\
+ 10540574310336x^8 + 20844113891396x^7 + 31429374388076x^6 \\
+ 36397721778208x^5 + 319893174321048x^4 + 206961055983144x^3 \\
+ 93048452882280x^2 + 25964710722144x + 3383816644368 \right) = 0.
\end{align*}\]

Galbraith has a small typo in his model for \(X_5^+ (157)\). His third equation should have leading coefficient \(2w^2\) instead of \(w^2\). The following linear map sends our model to his model

\[\begin{align*}
v & = 3V - W - X + Y + Z \\
w & = 2V - 2X - 2Y + Z \\
x & = -4V + 3X - Y - 3Z \\
y & = -V + X + Y - Z \\
z & = V - X + Z
\end{align*}\]
and for reference, the corrected Galbraith equations should be as follows:

\[-vx + 2wx - 5vy - 8vz + 5wz - 2xz + 5yz + z^2 = 0,
\]
\[w^2 + wx + 4wy - xy + y^2 + vz + 4wz - 2xz - 9yz - 12z^2 = 0,
\]
\[2w^2 + 3wx + x^2 - vy + 9wy + 2xy + 6y^2 + vz + 12wz + 5xz + 3yz - z^2 = 0.
\]

<table>
<thead>
<tr>
<th>$g = 5$, equations for canonical embedding in $\mathbb{P}^9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$XY + Y^2 - WZ + XZ + 3YZ + 4Z^2 = 0,$</td>
</tr>
<tr>
<td>$VW + 3WX + 3VY + WY + 4XY - 2Y^2 + VZ + 16WZ - 2XZ + 18YZ - 10Z^2 = 0,$</td>
</tr>
<tr>
<td>$VX + 3X^2 + VY - 6XY - 9Y^2 + 5VZ + 10WZ + 16XZ - 14YZ + 14Z^2 = 0,$</td>
</tr>
</tbody>
</table>
| $V^2W + VW^2 - W^3 + 3VWX + WX^2 - V^2X - V^2Y + VWY - 5W^2Y + 3VXY$
| $- 2WXY + 4X^2Y + VY^2 - 3VZ^2 - VWZ + 2W^2Z - 7VXZ - 15WXZ$
| $- 9X^2Z + 2VYZ = 2XYZ - 3XYZ - 8Y^2Z + VZ^2 + 4WZ^2 - XZ^2 + 4Y^2Z - 2Z^2 = 0$
| $V^2W + VW^2 - W^3 + V^2X + 3VWX + W^2X + 3V^2X^2 + X^3 + 2V^2Y + 3V^2Y$
| $- 7W^2Y + 8WXY - 6VY^2 - 7WY^2 + XY^2 - 2Y^3 + 4V^2Z + 14VWZ - 8W^2Z$
| $+ 14VXZ + 5X^2Z - 5VYZ - 4WYZ - XY^2 + 4Y^2Z + 6VZ^2 - 6WZ^2 + 3XZ^2 = 0$

with the variables corresponding to the following cusp forms:

$V = q + 2q^2 - 3q^3 - q^4 - q^5 - 5q^6 - 3q^7 - 2q^8 + 6q^9 - 3q^{10} - 6q^{11} + 2q^{12} + 5q^{13} - q^{14} + 3q^{15} - 4q^{16} - q^{17} + 8q^{18} + 2q^{19} + O(q^{20})$,

$W = 2q^2 - 5q^4 + 2q^5 - 2q^6 + 4q^7 + 4q^8 - 5q^{10} - q^{11} + 6q^{12} - 4q^{13} + 9q^{14} - 5q^{15} + 2q^{16} + q^{17} - 3q^{18} + 4q^{19} + O(q^{20})$,

$X = -4q^2 + 3q^3 + 3q^4 + 5q^5 + 4q^7 + q^8 - 6q^9 + 4q^{10} + q^{11} - 3q^{12} - 9q^{13} - 5q^{14} - 5q^{15} + 5q^{16} + 3q^{17} - 6q^{18} - 10q^{19} + O(q^{20})$,

$Y = -q^2 + 2q^4 + q^6 + q^7 - q^8 + 2q^{10} - 2q^{12} + q^{13} - 3q^{14} + q^{15} - q^{16} + q^{18} + 2q^{20}$

$Z = q^2 - q^5 - 2q^6 - q^7 + q^9 - q^{10} + q^{12} + 2q^{13} + 2q^{14} + q^{15} - q^{16} + q^{17} + q^{18} + 3q^{19} + O(q^{20})$

9 known rational points are:

<table>
<thead>
<tr>
<th>Cusp, $[1: 0: 0: 0: 0]$</th>
<th>$D = -16$, $[2: -2: -1: 1: 0]$</th>
</tr>
</thead>
</table>

For quadratic Chabauty, we use $p = 7$ and a single model $(\varphi_{181,1}$, $\mathcal{C}_{181,1})$

Information for $\varphi_{181,1}$:

$d_x = 3$

$d_y = 5$

$x_1 = Z$

$y_1 = (2 \cdot 17 \cdot 127)(V + W + X + 2Y + 2Z)$

$x_2 = X + Y + 8Z$

$y_2 = 2W - V + 3X - 6Y + 5Z$

Equation of $\mathcal{C}_{181,1}$ in the affine patch where $z = 1$ is:

$y^3 + (−11088x^5 + 16124x^4 - 9178x^3 + 2548x^2 - 344x + 18)y^2$

$+ \left(37937948x^{10} - 109104924x^9 + 139717096x^8 - 104840620x^7 + 51026716x^6 \right.$

$- 16825292x^5 + 3805736x^4 - 583156x^3 + 57968x^2 - 3380x + 88$

$\left. - 38931018912x^{15} + 165273816264x^{14} - 324830859688x^{13} + 391968536592x^{12} \right.$

$- 32467604314x^{11} + 195506522152x^{10} - 883896329808x^9 + 30555659584x^8$

$- 8141070384x^7 + 1672000152x^6 - 262592968x^5 + 30981584x^4$

$- 2659496x^3 + 156920x^2 - 5696x + 96 = 0.$
$g = 5$, equations for canonical embedding in $\mathbb{P}^{9}_{-1}$ are

\[
3VW + 2W^2 - VX + WX + X^2 + 3VY + 11WY + XY + 2Y^2 - 5VZ - 7WZ \\
- TXZ - 7YZ + 3Z^2 = 0, \\
VW + 3W^2 + VX + 2WX - X^2 - VY + 4XY - 3Y^2 - VZ + 2WZ - 8XZ + 13YZ - 12Z^2 = 0, \\
2V^2 + 3VW + 5W^2 + 3VX + WX - 4X^2 + 2VY - 7WY + 2XY + 2Y^2 - 2VZ \\
+ 12WZ - 10XZ - 7YZ + 3Z^2 = 0, \\
2W^3 + 2V^2X - VWX - 3WX^2 - 2X^3 - 2V^2Y - 4VXY - W^2Y - VXY - 6WXY \\
- X^2Y - VY^2 - Y^3 + 3VZ^2 - VWZ - 2W^2Z + 2VWX - 4WZX + 3VYZ - 6WYZ \\
- 3XYZ + 2Y^2Z - 3V^2Z - W^2Z + 5XZ^2 - YZ^2 - 2Z^3 = 0 \\
V^3 + V^2W + 5VW^2 + 2W^3 - VWX - W^2X - 2VX^2 - WX^2 + X^3 - 3VWY - W^2Y + VX + WXY - 2X^2Y - VY^2 - 3WX^2 - YX^2 - Y^3 - V^2Z + 3VWX + 8W^2Z \\
- VXZ - 4WXZ + 5X^2Z + VYZ + 6XYZ + 2Y^2Z + 3WZ^2 - 3XZ^2 + 4YZ^2 - 3Z^3 = 0
\]

with the variables corresponding to the following cusp forms

\[
V = -q + 2q^3 + q^7 - q^9 + 2q^{10} - q^{11} + 4q^{13} + 4q^{14} + 4q^{16} + 4q^{17} - 5q^{19} + O(q^{20}), \\
W = q^2 - q^5 - 2q^6 - 2q^7 - 2q^8 + 2q^{11} + 2q^{13} - q^{14} + 2q^{15} + q^{17} + q^{18} + 2q^{19} + O(q^{20}), \\
X = -q - q^3 + 2q^4 + 3q^5 + q^6 + 2q^7 - q^8 - q^{10} + 3q^{11} + 2q^{12} + 3q^{13} - q^{14} + 4q^{15} - 2q^{16} \\
- 3q^{17} + q^{18} + O(q^{20}), \\
Y = -q^2 + 2q^3 - q^5 + 3q^7 + 3q^8 + q^9 + 2q^{10} + q^{11} - 3q^{12} + q^{14} - 5q^{15} + q^{16} - q^{17} + q^{18} \\
+ 7q^{19} + O(q^{20}), \\
Z = q^3 - q^4 - q^5 - q^7 + 2q^8 + q^9 + q^{10} - q^{11} - 2q^{12} - q^{14} - 3q^{15} + q^{16} + q^{18} + 4q^{19} + O(q^{20})
\]

4 known rational points are

Cusp, [1: 0: 1: 0: 0] $D = -67, \ [1: 2: -1: -1: -1]$ \\
D = -8, [1: 0: 0: -1: -1] $D = -163, \ [1: -4: -7: 5: 2]$

For quadratic Chabauty, we use $p = 23$ and a single model $(\varphi_{227,1}, C_{227,1})$

Information for $\varphi_{227,1}$:

\[
d_x = 3 \\
x_1 = (7W + X - Y + 5Z) \\
x_2 = 2V + 6W + 5X - Y + 17Z
\]

Equation of $C_{227,1}$ in the affine patch where $z = 1$ is

\[
y^3 + \left( -16x^3 + 49x^2 - 41x + 5 \right) y^2 \\
+ \left( 102x^6 - 668x^5 + 1787x^4 - 2536x^3 + 2994x^2 - 1027x + 252 \right) y \\
+ \left( -289x^9 + 3111x^8 - 15134x^7 + 44025x^6 - 85025x^5 \right) + 113565x^4 - 104896x^3 + 64299x^2 - 23544x + 3888 = 0.
\]
$g = 5$, equations for canonical embedding in $\mathbb{P}^{9-1}$ are
\[
VW + 2X^2 + WY + 2XY - 2XZ - YZ + Z^2 = 0,
\]
\[
VX + 2X^2 + VY + WY + XY - VZ - 2XZ - YZ + Z^2 = 0,
\]
\[
WX - 3X^2 - VY - 2WY - 4XY - VZ + 4XZ + 3YZ - 2Z^2 = 0,
\]
with the variables corresponding to the following cusp forms
\[
V = q^6 - q^7 - q^8 + q^{11} - q^{12} + q^{13} + q^{16} + q^{17} - 2q^{18} + 2q^{19} + O(q^{20}),
\]
\[
W = -q + q^3 + q^5 + q^6 + q^7 + q^{10} + q^{12} + 3q^{13} + q^{14} - q^{15} + 2q^{16} + 2q^{17} - q^{18} + q^{19} + O(q^{20}),
\]
\[
X = q^4 - q^5 - q^6 + q^9 + q^{11} - q^{12} - q^{13} + q^{15} - 3q^{16} + 2q^{17} + q^{18} + O(q^{20}),
\]
\[
Y = q^3 - q^4 - q^7 - 2q^9 + q^{10} - q^{11} + q^{12} + q^{13} - q^{15} + 2q^{16} + 2q^{17} + q^{19} + O(q^{20}),
\]
\[
Z = q^2 - q^5 - 2q^6 - q^7 - q^8 - q^{10} + q^{13} - q^{14} + 2q^{15} - q^{16} + q^{17} + 2q^{18} + 2q^{19} + O(q^{20})
\]
6 known rational points are
\[
\text{Cusp, } [0: 1: 0: 0: 0] \quad D = -19, \quad [0: 0: 0: 1: 0] \quad D = -67, \quad [1: 0: 1: -1: 1] \quad D = -163, \quad [2: 1: 3: -4: 4]
\]
For quadratic Chabauty, we use $p = 23$ and a single model ($\phi_{263,1}$,$C_{263,1}$)
Information for $\phi_{263,1}$:
\[
d_x = 6 \quad d_y = 6
\]
\[
x_1 = V - 2X + Z \quad y_1 = W + X - Z
\]
\[
x_2 = 2V - W - X \quad y_2 = 2W - W - X
\]
Equation of $C_{263,1}$ in the affine patch where $z = 1$ is
\[
y^6 + \frac{1}{2} (-4x + 9) y^5 + \frac{1}{2} (7x^2 - 24x + 24) y^4 + \frac{1}{2} (-6x^3 + 19x^2 - 28x + 28) y^3 + \frac{1}{2} (7x^4 - 31x^3 + 41x^2 - 19x + 13) y^2 + \frac{1}{2} (-2x^5 + 9x^4 - 21x^3 + 25x^2 - 9x + 2) y + \frac{1}{2} (2x^6 - 9x^5 + 14x^4 - 9x^3 + 4x^2 - x) = 0.
\]
5.3 Genus 6

$$g = 6,$$ equations for canonical embedding in $$\mathbb{P}^{52}$$ are

$$V X - U Y - X Y = 0,$$
$$U Y - W Y - V Z = 0,$$
$$U X - W X - U Z - X Z = 0,$$
$$U X + V X + V Y + V Z - Y Z - Z^2 = 0,$$
$$U V + V^2 - V Y + W Y + W Z + X Z = 0,$$
$$U^2 + U V - V W + W X + V Y + X Y = 0$$

with the variables corresponding to the following cusp forms

$$U = -q^2 + q^3 + q^4 - q^9 + 3q^{10} - 2q^{11} + 2q^{12} + 2q^{14} - 4q^{15} - 2q^{18} + q^{19} + O(q^{20}),$$
$$V = -q^3 + q^4 + q^5 + 2q^6 - 2q^{10} + q^{11} - 2q^{12} - 3q^{13} - q^{14} + 3q^{15} - 3q^{16} - q^{17} + 3q^{18} + 4q^{19} + O(q^{20}),$$
$$W = -q + q^3 + q^4 + 2q^5 + q^7 + q^8 + q^{10} + q^{12} + q^{14} - 2q^{15} - 3q^{16} + 3q^{17} - q^{18} + 2q^{19} + O(q^{20}),$$
$$X = -q^5 + q^6 + 2q^7 - q^8 - 2q^{11} - 2q^{12} + q^{13} - q^{14} + 3q^{15} + q^{17} - 2q^{18} - 4q^{19} + O(q^{20}),$$
$$Y = -q^6 + q^7 - 2q^8 - q^{10} - 3q^{11} + q^{12} - q^{13} + q^{14} + q^{15} - 5q^{16} + q^{17} + 4q^{18} + 4q^{19} + O(q^{20}),$$
$$Z = q^4 - q^5 - q^6 - 2q^8 + 2q^9 + q^{10} + q^{11} + q^{14} + 2q^{16} - q^{18} - q^{19} + O(q^{20})$$

11 known rational points are

\[
\begin{align*}
\text{Cusp}, & \quad [0: 0: 1: 0: 0: 0] & D = 19, & \quad [1: -1: 0: 0: 0: 0] \\
D = -3, & \quad [0: 1: -2: 3: 1: 2] & D = 27, & \quad [0: 1: 1: 1: 0: 1: 1] \\
D = -7, & \quad [0: 0: 0: 0: 1: -1] & D = 67, & \quad [1: 0: -1: 1: 0: 1] \\
D = -8, & \quad [0: 0: 0: 1: 0: 0] & D = 28, & \quad [2: -2: 0: 2: -1: 1] \\
D = -11, & \quad [0: 0: 0: 0: 1: 0] & D = 163, & \quad [78: -50: -42: -13: 10: -24] \\
D = -12, & \quad [0: 1: 0: -1: 1: 0] & \quad \text{For quadratic Chabauty, we use } p = 31 \text{ and a single model } (\varphi_{163,1}, C_{163,1}) \\
\text{Information for } \varphi_{163,1}: & \quad \text{ Equation of } C_{163,1} \text{ in the affine patch where } z = 1 \text{ is} \\
\frac{d_x}{d_y} = 4 & \quad \text{ Equation of } C_{163,1} \text{ in the affine patch where } z = 1 \text{ is} \\
x_1 = Y + Z & \quad \frac{d_x}{d_y} = 4 & \quad \text{d}_y = 4 \\
x_2 = 2X - Y - Z & \quad y_1 = V - Y \\
x_2 = 2X - Y - Z & \quad y_2 = V + Y \\
\text{Equation of } C_{163,1} \text{ in the affine patch where } z = 1 \text{ is} & \quad \frac{y^4 - \frac{1}{2^{11} \cdot 5^4} \left(x^4 + 16 x^3 + 22 x^2 + 8 x + 1\right) y^3}{y^3} \\
- \frac{1}{2^6 \cdot 5^2} \left(2920x^8 + 9243x^7 + 13419x^6 + 11947x^5 + 7067x^4 + 2873x^3 + 777x^2 + 129x + 9\right) y^2 \\
+ \frac{1}{2^{9} \cdot 5^3} \left(52800x^{12} + 211440x^{11} + 414553x^{10} + 528350x^9 + 484599x^8 + 337200x^7\right) y \\
- \frac{1}{2^{12} \cdot 5^4} \left(960000x^{16} + 3242000x^{15} + 5184200x^{14} + 4513325x^{13} + 1189308x^{12}\right) \left(-246947x^{11} - 4252380x^{10} - 3906469x^9 - 2536840x^8 - 1251364x^7\right) \left(-479056x^6 - 142093x^5 - 31924x^4 - 5166x^3 - 540x^2 - 27x\right) &= 0.
\end{align*}
\]
For quadratic Chabauty, we use $C_{0}: [0: 0: 0: 0: 0: 1]$ with the variables corresponding to the following cusp forms

$$
U = -q^3 + q^4 + q^5 - q^6 + q^9 - q^{10} + 2q^{11} - q^{12} + q^{13} - q^{14} + q^{15} - q^{16} - 4q^{17} + 2q^{18} + 2q^{19} + O(q^{20}),
$$

$$
V = -q^2 + q^4 + q^5 + 2q^6 + q^{11} - q^{12} - q^{13} + 2q^{14} - 2q^{15} - q^{16} - q^{17} - 3q^{18} + O(q^{20}),
$$

$$
W = q^4 - q^5 - q^6 - q^{10} + 3q^{11} - q^{12} + 3q^{15} - 2q^{16} - 2q^{17} + 4q^{18} - q^{19} + O(q^{20}),
$$

$$
X = q^5 + q^6 + q^7 + q^9 - q^{10} - 2q^{12} - q^{13} + 4q^{17} - 4q^{18} - 2q^{19} + O(q^{20}),
$$

$$
Y = q^7 + q^8 + 2q^9 - 2q^{11} - 2q^{12} + 2q^{13} - q^{14} - q^{15} + 3q^{17} - q^{18} - q^{19} + O(q^{20}),
$$

$$
Z = q - q^3 - q^4 - q^5 - q^7 - q^8 - q^{10} - 2q^{11} - 3q^{13} + q^{14} - q^{15} + q^{16} + 2q^{17} - 2q^{18} - 3q^{19} + O(q^{20})
$$

8 known rational points are

$$
\begin{align*}
\text{Cusp, [0: 0: 0: 0: 0: 1]} & \quad D = -19, [0: 0: 0: 0: 1: 0] \\
D = -7, [1: 0: 1: -1: 1: -1] & \quad D = -43, [0: 0: 1: 0: -1: 1] \\
\end{align*}
$$

For quadratic Chabauty, we use $p = 23$ on two patches of the same projective model $\{(\varphi_{197,i}, C_{197,i})\}_{i=1}^{2}$. Information for $\varphi_{197,1}$:

$$
\begin{align*}
d_x = 6 & \quad d_y = 6 \\
x_1 = U + Z & \quad y_1 = V + W + Z \\
x_2 = 2X - 2Y & \quad y_2 = X - Y
\end{align*}
$$

Information for $\varphi_{197,2}$:

$$
\begin{align*}
d_x = 5 & \quad d_y = 6 \\
x_1 = U + Z & \quad y_1 = X - Y \\
x_2 = V + W + Z & \quad y_2 = V + W + Z
\end{align*}
$$

Equation of $C_{197,1}$ in the affine patch where $z = 1$ is

$$
\begin{align*}
y^6 - 10xy^5 \\
+ (44x^2 + 10x - 3)y^4 \\
+ (-112x^3 - 48x^2 + 2x + 2)y^3 \\
+ (176x^4 + 112x^3 + 16x^2 - 4x + 2)y^2 \\
+ (-160x^5 - 128x^4 - 40x^3 - 2x - 2)y \\
+ (64x^6 + 64x^5 + 32x^4 + 2x) = 0.
\end{align*}
$$

Equation of $C_{197,2}$ in the affine patch where $z = 1$ is

$$
\begin{align*}
y^5 + (-x + 2)y^4 \\
+ (-2x^2 + 6x - 4)y^3 \\
+ (2x^6 - 13x^5 + 32x^4 - 35x^3 + 9x^2 + 16x - 12)y^2 \\
+ (2x^8 - 20x^7 + 86x^6 - 208x^5 + 309x^4 - 286x^3 + 156x^2 - 40x)y \\
+ (x^{10} - 13x^9 + 75x^8 - 254x^7 + 563x^6 - 861x^5 + 929x^4 - 704x^3 + 360x^2 - 112x + 16) = 0.
\end{align*}
$$
\( g = 6 \), equations for canonical embedding in \( \mathbb{P}^{\delta - 1} \) are

\[
\begin{align*}
UV - V^2 + X^2 - UY + VY &= 0, \\
W^2 - VX - WX + WY - XY + YZ &= 0, \\
UV + VX + WX - UY - VZ + YZ &= 0, \\
VW + UX - VX - WX + X^2 - WY + YZ &= 0, \\
UW - VX - WX + UY - VY - WY + XZ + YZ &= 0, \\
U^2 - UW + UX - WX - VY - VZ + WZ + YZ &= 0,
\end{align*}
\]

with the variables corresponding to the following cusp forms

\[
\begin{align*}
U &= -q^2 + q^3 + q^4 - q^7 - q^9 + 2q^{10} + 2q^{12} + 2q^{14} - 2q^{15} - q^{16} - q^{17} + q^{19} + O(q^{20}), \\
V &= q^5 - 2q^7 - q^8 - q^9 + 2q^{11} + 2q^{12} + q^{13} + q^{14} - q^{18} + O(q^{20}), \\
W &= q^3 - q^5 + q^6 - q^9 + q^{10} - q^{13} - 3q^{15} + q^{16} + 2q^{17} - q^{18} + q^{19} + O(q^{20}), \\
X &= -q^4 + q^6 - q^9 + q^{10} + q^{12} - 2q^{13} - q^{14} - 2q^{15} + 2q^{16} + q^{17} - 2q^{18} - q^{19} + O(q^{20}), \\
Y &= q^5 - q^6 - q^7 - q^{11} + 2q^{12} + 3q^{14} - 3q^{17} + q^{18} + q^{19} + O(q^{20}), \\
Z &= -q + 3q^2 + 2q^5 + q^6 + q^8 + q^{10} + 2q^{11} + q^{12} + q^{13} + q^{14} - 2q^{15} + 2q^{17} - 2q^{18} + q^{19} + O(q^{20}),
\end{align*}
\]

\( X_0^+(211) \)

8 known rational points are

| Cusp, \( [0: 0: 0: 0: 0: 1] \) | \( D = -12 \), \( [1: 1: 0: 0: 0: 1] \) |
| \( D = -3 \), \( [1: -3: 2: 2: -2: 3] \) | \( D = -27 \), \( [1: 0: -1: -1: 1: 0] \) |
| \( D = -7 \), \( [1: 1: 0: 0: 1: 0] \) | \( D = -28 \), \( [1: -1: 0: -2: 1: 0] \) |
| \( D = -8 \), \( [0: 0: 0: 0: 1: 0] \) | \( D = -67 \), \( [1: 0: 1: 1: 0] \) |

For quadratic Chabauty, we use \( p = 31 \) and a two models \( \{ (\varphi_{211,i}, C_{211,1}) \}_{i=1}^2 \)

Information for \( \varphi_{211,1} \):

\[
\begin{align*}
d_x &= 6, & d_y &= 6 \\
x_1 &= X - Y - Z, & y_1 &= U + 2Y + Z \\
x_2 &= V - W - 2Y, & y_2 &= V - W - 2Y
\end{align*}
\]

Information for \( \varphi_{211,2} \):

\[
\begin{align*}
d_x &= 6, & d_y &= 6 \\
x_1 &= U + 2Y + Z, & y_1 &= V - W - 2Y \\
x_2 &= X - Y - Z, & y_2 &= X - Y - Z
\end{align*}
\]

followed by automorphism \([-C, B + C, A + 2C] : \mathbb{P}^2 \to \mathbb{P}^2\).

Equation of \( C_{211,1} \) in the affine patch where \( z = 1 \) is

\[
\begin{align*}
y^6 + (9x + 1)y^5 + (32x^2 + 12x - 2)y^4 + (57x^3 + 49x^2 - 6x - 4)y^3 \\
+ (53x^4 + 89x^3 + 5x^2 - 19x - 1)y^2 + (24x^5 + 72x^4 + 30x^3 - 26x^2 - 8x + 2)y \\
+ 4x^6 + 20x^5 + 24x^4 - 8x^3 - 14x^2 + 3x + 1
\end{align*}
\]

Equation of \( C_{211,2} \) in the affine patch where \( z = 1 \) is

\[
\begin{align*}
y^6 + (2 + 7x)y^5 + (-1 + 14x + 18x^2)y^4 + (-4 - 9x + 38x^2 + 22x^3)y^3 \\
+ (-2 - 22x - 28x^2 + 48x^3 + 13x^4)y^2 + (1 - 6x - 39x^2 - 36x^3 + 28x^4 + 3x^5)y \\
+ 1 + 4x - 2x^2 - 22x^3 - 17x^4 + 6x^5
\end{align*}
\]

with \( \mathbb{F}_p \)-points \( [-1: 1: 1], [-1/2: 0: 1], [1: -2: 1], [-2: 3: 1], [-1: 2: 1], [0: -1: 1] \) on the affine patch with \( z = 1 \) and infinite points \( [-1/3: 1: 0], [1: 0: 0] \)
For quadratic Chabauty, we use the cusp forms
\begin{align*}
U &= q^6 - q^7 - q^8 - q^{10} + 2q^{11} - q^{12} + q^{14} + 3q^{16} - 2q^{18} - 2q^{19} + O(q^{20}), \\
V &= -q^5 + q^6 + q^7 - 2q^{12} + 2q^{13} - 2q^{14} - q^{16} + q^{17} - q^{18} - q^{19} + O(q^{20}), \\
W &= -q^2 + q^3 + q^4 - q^7 - q^9 + q^{10} + q^{11} + q^{12} + 2q^{14} - q^{15} + 3q^{16} + q^{17} - 3q^{19} + O(q^{20}), \\
X &= q^3 - q^4 - q^6 - q^7 + 2q^8 - 2q^9 + 2q^{12} - q^{13} + q^{14} - q^{15} + 2q^{17} + 2q^{18} - 2q^{19} + O(q^{20}), \\
Y &= -q + q^2 + q^5 + q^7 + 2q^9 - q^{10} + q^{11} + 2q^{13} + q^{15} - q^{16} + 2q^{17} - q^{18} + q^{19} + O(q^{20}), \\
Z &= -q^2 + q^4 + q^5 + q^6 + q^8 + q^{10} + q^{11} - q^{12} - 2q^{13} - q^{19} + O(q^{20}).
\end{align*}

7 known rational points are
\begin{align*}
\text{Cusp, } [0: 0: 0: 0: 1: 0] & \quad D = -27, [0: 1: -1: -1: 1: 0] \\
\text{D = -11, } [0: 0: 1: 1: 0: 0] & \quad D = -163, [2: -2: 1: 3: -4: -6] \\
\text{D = -12, } [1: 0: 1: 1: 0: 0] & \quad D = -32, [0: 0: 1: 1: 0: 0]
\end{align*}

For quadratic Chabauty, we use \( p = 19 \) on two patches of the same projective model \( \{(\varphi_{223,1}, C_{223,1})\}_{i=1} \).

Information for \( \varphi_{223,1} \):
\begin{align*}
d_x &= 6 & d_y &= 5 \\
x_1 &= U - W & y_1 &= 2(V - Y) \\
x_2 &= X + Y & y_2 &= X + Y
\end{align*}

Information for \( \varphi_{223,2} \):
\begin{align*}
d_x &= 6 & d_y &= 5 \\
x_1 &= X + Y & y_1 &= 2(V - Y) \\
x_2 &= U - W & y_2 &= U - W
\end{align*}

Equation of \( C_{223,1} \) in the affine patch where \( z = 1 \) is
\begin{align*}
y^6 + (-x + 8)y^5 \\
+ (8x^2 + 30)y^4 \\
+ (-4x^3 + 88x^2 + 40x + 60)y^3 \\
+ (8x^4 - 32x^3 + 264x^2 + 128x + 64)y^2 \\
+ (-16x^5 - 192x^3 + 256x^2 + 176x + 32)y \\
+ (-96x^5 - 128x^4 - 160x^3 + 128x) = 0.
\end{align*}

Equation of \( C_{223,2} \) in the affine patch where \( z = 1 \) is
\begin{align*}
y^6 + (8x - 1)y^5 \\
+ (30x^2 + 8)y^4 \\
+ (60x^3 + 40x^2 + 88x - 4)y^3 \\
+ (64x^4 + 128x^3 + 264x^2 - 32x + 8)y^2 \\
+ (32x^5 + 176x^4 + 256x^3 - 192x^2 - 16)y \\
+ (128x^5 - 160x^3 - 128x^2 - 96x) = 0.
\end{align*}
For quadratic Chabauty, we use 6 known rational points are
\[ W = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{2} = 1 \]
with the variables corresponding to the following cusp forms
\[
U = -q^3 + q^6 + q^8 - q^9 + q^{10} + q^{11} - q^{13} + 2q^{14} - 2q^{15} + q^{16} - q^{17} - q^{19} + O(q^{20}),
\]
\[
V = -q^3 + q^6 + q^7 + q^9 - q^{10} - q^{11} + q^{12} - q^{16} - 2q^{18} + q^{19} + O(q^{20}),
\]
\[
W = -q^7 + q^8 - q^9 + q^{12} + 2q^{13} - 2q^{14} + q^{17} - q^{19} + O(q^{20}),
\]
\[
X = -q^3 + 2q^4 + q^7 + q^{11} + 3q^{14} - 2q^{15} + 2q^{17} - 2q^{19} + O(q^{20}),
\]
\[
Y = q^2 - q^4 - q^5 - q^6 + q^9 + q^{12} - 2q^{14} + q^{16} - q^{17} - q^{18} + O(q^{20}),
\]
\[
Z = 3q^3 + q^4 + q^7 + q^{10} + 3q^{11} + q^{13} + q^{15} + q^{17} + 2q^{19} + O(q^{20})
\]

For quadratic Chabauty, we use \( p = 29 \) and a single model \( (\varphi_{269,1}, C_{269,1}) \)

Information for \( \varphi_{269,1} \):

\[
\begin{align*}
&d_x = 4 \\
&d_y = 6 \\
x_1 = 2Y - 2Z \\
x_2 = 2Y - Z \\
y_1 = U \\
y_2 = W - Y + 2Z
\end{align*}
\]

Equation of \( C_{269,1} \) in the affine patch where \( z = 1 \) is

\[
y^4 + \frac{1}{3 \cdot 11} \left( 121x^6 - 1045x^5 + 3614x^4 - 6388x^3 + 6048x^2 - 2768x + 320 \right) y^3
\]
\[
+ \frac{1}{3^2 \cdot 11^2} \left( 2145x^{12} - 5444x^{11} + 276793x^{10} - 630098x^9 + 790076x^8 - 845722x^7 + 1657072x^6 \right)
\]
\[
- 3337920x^5 + 4271616x^4 - 3294464x^3 + 1493248x^2 - 378880x + 53248
\]
\[
+ \frac{1}{3^3 \cdot 11^3} \left( -47916x^{18} + 2646270x^{17} - 5453995x^{16} + 55774407x^{15} - 347908604x^{14} + 1467762288x^{13} \right)
\]
\[
- 4439349260x^{12} + 99794532256x^{11} - 170613685632x^{10} + 22502856064x^9
\]
\[
- 230718110592x^8 + 184336004352x^7 - 11451136768x^6 + 5494525328x^5
\]
\[
- 20124897280x^4 + 5510791168x^3 - 1082523648x^2 + 1394606068x - 9437184
\]
\[
+ \frac{1}{3^4 \cdot 11^4} \left( -1006236x^{24} + 76591548x^{23} - 2333162502x^{22} + 37125740826x^{21} - 348516546390x^{20} \right.
\]
\[
+ 2155553249634x^{19} - 9476259572816x^{18} + 31140876323216x^{17} - 7913312805312x^{16}
\]
\[
+ 159133059208064x^{15} - 257270458757504x^{14} + 337929123632384x^{13} - 363041352011776x^{12}
\]
\[
+ 32016124717344x^{11} - 232066303220064x^{10} + 138148922114048x^9 - 67352475975680x^8
\]
\[
+ 26747617058816x^7 - 8571322695680x^6 + 2180924702720x^5 - 428701908992x^4
\]
\[
+ 62014881792x^3 - 5989466112x^2 + 301989888x \right) = 0.
\]
$g = 6$, equations for canonical embedding in $\mathbb{P}^{9-1}$ are

\begin{align*}
UV + VX - WX - WY + VZ & = 0, \\
U^2 + UV - UW - UY - XY - Z^2 & = 0, \\
UV + V^2 + UX + UY + XY + Y^2 + VZ + 2WZ & = 0, \\
U^2 - VX - XY - Y^2 + UZ - VZ + XZ + YZ & = 0, \\
VW - VX + UY + VY + XY + Y^2 + UZ + XZ + YZ & = 0, \\
UV - VW - UX - VX - UY - 2VY - Y^2 - VZ - WZ + XZ & = 0,
\end{align*}

with the variables corresponding to the following cusp forms

\begin{align*}
U & = -q^3 + q^5 + q^6 + q^8 + q^9 + q^{11} - q^{12} - 2q^{14} + q^{15} - 3q^{16} - q^{17} - q^{18} + O(q^{20}), \\
V & = q^4 - q^5 - 2q^8 + q^{11} + 2q^{13} - q^{14} + q^{15} + O(q^{20}), \\
W & = -q + q^2 + q^5 + q^9 - q^{10} + 3q^{11} + q^{13} + q^{15} - q^{16} + 2q^{17} + q^{19} + O(q^{20}), \\
X & = q^2 - q^4 - q^5 - q^8 - q^9 - 2q^{10} + q^{11} + q^{13} - q^{18} + O(q^{20}), \\
Y & = -q^2 + q^4 + q^6 + q^7 + q^8 + q^9 + q^{10} - q^{11} - q^{12} + q^{13} - q^{16} - q^{17} + q^{19} + O(q^{20}), \\
Z & = q^7 - q^8 - q^9 + q^{12} - q^{13} + 3q^{16} + q^{17} + q^{18} - 2q^{19} + O(q^{20})
\end{align*}

6 known rational points are

\begin{align*}
\text{Cusp, } [0: 0: 1: 0: 0: 0] & \quad D = -19, [0: 0: 0: 1: 0: 0] \\
D = -12, [1: 0: 0: -1: 1: -1] & \quad D = -43, [0: 0: 0: 1: -1: 1]
\end{align*}

For quadratic Chabauty, we use $p = 13$ and a single model $(\varphi_{271,1}, C_{271,1})$

Information for $\varphi_{271,1}$:

\begin{align*}
d_x & = 5, & d_y & = 9, \\
x_1 & = V + W, & y_1 & = 80(2W - 2X - Y + Z), \\
x_2 & = U - W + 2X + Y, & y_2 & = U - W + 2X + Y
\end{align*}

Equation of $C_{271,1}$ in the affine patch where $z = 1$ is

\begin{align*}
g^5 + (-85x^5 + 1127x^4 - 1913x^3 - 1182x^2 - 163x - 6)y^4 \\
+ \left(10080x^{10} - 140412x^9 + 690654x^8 - 1381518x^7 + 483107x^6 + 1493128x^5\right)g^3 \\
+ \left(629335x^4 + 89925x^3 + 672x^2 - 820x - 48\right)g^2 \\
+ \left(51200x^{15} + 5322240x^{14} - 62218208x^{13} + 265115488x^{12} - 497719884x^{11}\right)g \\
+ \left(+393876380x^{10} + 397941188x^9 - 406993184x^8 - 419868463x^7 - 107094101x^6\right) \\
+ \left(+10069328x^5 + 10921601x^4 + 2495279x^3 + 280582x^2 + 16084x + 376\right) \\
+ \left(-77824000x^{20} + 842444800x^{19} - 1778347520x^{18} - 10376567744x^{17} + 5718517908x^{16}\right) \\
+ \left(-105874923384x^{15} + 8248358348x^{14} + 3704038788x^{13} - 128890371748x^{12} - 976838538x^{11}\right) \\
+ \left(+103912556718x^{10} + 65061175451x^9 + 8351258644x^8 - 6617560964x^7 - 377123352x^6\right) \\
+ \left(-97933013x^5 - 153099583x^4 - 15265548x^3 - 951824x^2 - 33920x - 528\right) \\
+ \left(1966080000x^{25} - 32800768000x^{24} + 19565690800x^{23} - 384548372480x^{22} - 835604847872x^{21} + 538540280384x^{20} - 1025107034808x^{19} + 722206561294x^{18}\right) \\
+ \left(+427168711728x^{17} - 1045673369424x^{16} + 2541993008328x^{15} + 7501031432496x^{14}\right) \\
+ \left(-489981838384x^{13} + 10352281169140x^{12} - 5911510862587x^{11} - 1394541574295x^{10}\right) \\
+ \left(+91202017428x^9 + 165638010722x^8 + 56008509377x^7 + 10935753977x^6\right) \\
+ \left(+1399973446x^5 + 119863400x^4 + 6648960x^3 + 216912x^2 + 3168x\right)
\end{align*}
\(g = 6\), equations for canonical embedding in \(\mathbb{P}^{9 + 1}\) are
\[
UV + VX + VY + UZ + VZ + WZ = 0,
\]
\[
UV + 2UW + VW + W^2 + WY + WZ + XZ = 0,
\]
\[
U^2 - V^2 - VW - UX + UY + WY + Y^2 = 0,
\]
\[
V^2 + VW + UY + VY + UZ + VZ - XZ + YZ = 0,
\]
\[
U^2 + UW - UX - WX - WY + UZ + WZ - XZ = 0,
\]
\[
UV + UW + VW - VX - WX - UY - WY - Y^2 - YZ = 0,
\]
with the variables corresponding to the following cusp forms
\[
U = -q^3 + q^5 + q^7 + q^9 + q^{10} + q^{12} - q^{13} - q^{16} - q^{17} + O(q^{20}),
\]
\[
V = q^8 - q^9 - q^{10} + q^{13} + q^{15} - q^{16} + q^{19} + O(q^{20}),
\]
\[
W = q^3 - q^6 - q^7 - q^8 - q^9 - q^{11} - q^{13} + q^{14} - q^{15} + 2q^{16} + q^{17} + 2q^{18} + q^{19} + O(q^{20}),
\]
\[
X = -q + q^4 + q^5 + q^7 + q^9 + q^{10} + q^{11} + q^{12} + q^{13} + q^{15} + 2q^{17} + 2q^{19} + O(q^{20}),
\]
\[
Y = q^2 - q^4 - q^5 - q^{10} - q^{12} + q^{13} - q^{14} - q^{17} - 2q^{18} + O(q^{20}),
\]
\[
Z = q^4 - q^5 - q^6 - q^8 - q^{10} + q^{11} + q^{15} - q^{16} + 2q^{17} - q^{18} - q^{19} + O(q^{20})
\]
7 known rational points are
\[\text{Cusp, } [0: 0: 0: 1: 0: 0] \quad D = -43, [1: 0: 0: 1: 0: 0]
\]
\[D = -7, [0: 1: 0: 0: 1: -1] \quad D = -67, [1: 0: -2: 1: 0: 0]
\]
\[D = -19, [0: 0: 0: 0: 0: 1] \quad D = -163, [0: 2: -6: 6: 4: 5]
\]
\[D = -28, [0: 1: 0: 0: -1: 1]
\]
For quadratic Chabauty, we use \(p = 7\) and two patches of the same projective model \(\{(\varphi_{359,i}, C_{359,i})\}_{i=1}^{2}\).

Information for \(\varphi_{359,1}\):
\[
d_x = 6 \quad d_y = 6
\]
\[
x_1 = W + Y \quad y_1 = U + V
\]
\[
x_2 = X \quad y_2 = X
\]

Information for \(\varphi_{359,2}\):
\[
d_x = 6 \quad d_y = 5
\]
\[
x_1 = X \quad y_1 = W + Y
\]
\[
x_2 = U + V \quad y_2 = U + V
\]

Equation of \(C_{359,1}\) in the affine patch where \(z = 1\) is
\[
y^6 + (2x + 1)y^5 + (4x^2 - 10x - 6)y^4 + (2x^3 - 15x^2 - 7x + 9)y^3 + (-4x^4 - 10x^3 - 9x^2 + 12x - 6)y^2 + (-4x^5 - 4x^4 - 6x^3 + 8x^2 - 3x + 1)y + (-x^6 - x^5 - 2x^4 + x^3 - x^2) = 0.
\]

Equation of \(C_{359,2}\) in the affine patch where \(z = 1\) is
\[
y^6 + (x + 4)y^5 + (2x^2 + 4x + 4)y^4 + (-x^3 + 6x^2 + 10x - 2)y^3 + (x^4 - 8x^3 + 9x^2 + 15x - 4)y^2 + (3x^4 - 12x^3 + 7x^2 + 10x - 2)y + (-x^5 + 6x^4 - 9x^3 + 6x^2 - x - 1) = 0.
\]
References


