Some Discussions on Restrictions of Gaussian Free Fields and Massive Gaussian Free Fields

Linan Chen

May 15th, 2009

Introduction:

Suppose$(\Omega, \mathcal{F}, \mathbb{P})$ is a general probability space, a Gaussian Free Field (GFF) is a family of centered Gaussian variables, closed under $L^2(\Omega, \mathcal{F}, \mathbb{P})$, indexed by some Hilbert space H, denoted as $\{(h, g)_H : g \in H\}$ and with the covariance given by

$$\mathbb{E}\left[(h, g_1)_H (h, g_2)_H\right] = (g_1, g_2)_H .$$

We are interested in two kinds of GFF. The first one, mainly referred as GFF in the forth comings, is indexed by $H(D)$, where $D \subseteq \mathbb{R}^n$ is some bounded domain, and $H(D)$ is the closure of $C_c^\infty(D)$ under the Dirichlet inner product $(\cdot, \cdot)_\nabla$ on D. The other one is the massive GFF. which is indexed by the standard Sobolev space $H^1(\mathbb{R}^n)$.

As we can see from [3] and [4], when $n = 2$, the circle "average" of the GFF has the same law as a multiple of Brownian motion. Inspired by this, we discussed the sphere "average" of the GFF for $n \geq 3$, and also the restrictions of the GFF on general curves ($n = 2$) and hypersurfaces ($n \geq 3$).

We also studied the massive GFF, mainly with the approach of Fourier analysis. By examining the decay rate of the Fourier transform of the restriction distribution, we got the conclusion that the massive GFF can be restricted on any smooth compact hypersurface with non-zero Gaussian curvature everywhere.

S^{n-1} Average of GFF:

Consider a GFF indexed by $H(D)$ where $D = B(0, 1)$, and denote h to be an instance of the GFF. As mentioned in the previous section, for any $x_0 \in D$, when h is restricted on concentric circles $\left\{S^1(x_0, e^{-t}) : t > t_0\right\}$ for some large $t_0 > 0$, the t -parametrized "restriction" process, after renormalization, has the same law of a multiple of a Brownian motion starting from some point $c(x_0) \in \mathbb{R}$. This can be seen from direct computations involving the Green function $G_D(x, y)$ of Laplace operator on the domain D.

Without loss of generality, we can assume $x_0 = 0$, denote $S_t = S^1(0, e^{-t})$, and χ_t the uniform measure on S_t. Consider the action (h, χ_t) as taking the average of h on S_t in the distribution sense. According to the characterization of GFF, the covariance of the arising one-parameter process $\{(h, \chi_t) : t > 0\}$ is given by

$$\frac{e^{t+s}}{4\pi^2} \int_{S_t} \int_{S_s} G_D(x, y)\, d\sigma(x)\, d\sigma(y),$$

where $0 < t, s < \infty$, and $d\sigma$ is the standard length measure on S_t. Notice that in this case, $G_D(x, y)$ is explicitly given by

$$G_D(x, y) = -\frac{1}{2\pi}\left(\log|x-y| - \log\left||x|\, y - \frac{x}{|x|}\right|\right),$$

for $x, y \in D$. With direct computations, one sees that

$$\frac{e^{t+s}}{4\pi^2} \int_{S_t} \int_{S_s} G_D(x, y)\, d\sigma(x)\, d\sigma(y) = \frac{t \wedge s}{2\pi}.$$

We want to study the similar property of GFF whose instance h takes value in higher dimensional space. To this end, we consider $D = B(0, 1) \subseteq \mathbb{R}^n$ and the GFF indexed by $H(D)$ with instance h. Again, we fix $x_0 = 0$, and denote χ_r as the uniform measure on $S^{n-1}(0, r)$ for $0 < r < 1$. Similarly, we can compute the covariance of (h, χ_{r_1}) and (h, χ_{r_2}) for $0 < r_1 < r_2 < 1$ as

$$\frac{1}{\sigma_n^2}(r_1 r_2)^{1-n} \int_{S^{n-1}(0, r_1)} \int_{S^{n-1}(0, r_2)} G_D(x, y)\, d\sigma(x)\, d\sigma(y),$$

where σ_n is the surface area of the unit sphere in $\mathbb{R}^n$, and $d\sigma$ is the standard surface measure on the sphere. Since the Green function on D is given by

$$G_D(x, y) = \frac{1}{(n-2)\sigma_n}\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{\left||x|\, y - \frac{x}{|x|}\right|^{n-2}}\right),$$

we first evaluate the following integral,

$$\int_{S^{n-1}(0, r_1)} \int_{S^{n-1}(0, r_2)} \frac{1}{|x-y|^{n-2}}\, d\sigma(x)\, d\sigma(y).$$

By the rotational symmetry of the two spheres, we can fix $z_0 = (0, \cdots, r_1) \in S^{n-1}(0, r_1)$, then the integral above is equal to

$$\begin{aligned}
& \sigma_n r_1^{n-1} \int_{S^{n-1}(0, r_2)} \frac{1}{|x-z_0|^{n-2}}\, d\sigma(x) \\
= \; & \sigma_n (r_1 r_2)^{n-1} \int_0^{\pi} \frac{\sin^{n-2}\theta}{(r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta)^{\frac{n-2}{2}}}\, d\theta \\
= \; & \sigma_n (r_1 r_2)^{n-1} \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right) r_2^{2-n}}{\Gamma\left(\frac{n}{2}\right)}.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[\left(h,\,\chi_{r_1}\right)\left(h,\,\chi_{r_2}\right)\right]=c_n r_2^{2-n}+c_0,$$

where c_n is a constant only depending on the dimension and c_0 is a constant independent of r_2, r_1 or n. Thus, in n-dimensional space, $n \geq 3$, for any $x_0 \in D$, the average of GFF on the concentric S^{n-1}spheres $\left\{S^{n-1}\left(x_0,\,t^{\frac{1}{2-n}}\right):\,t>t_0\right\}$ for some large enough $t_0>0$, has the same law of a multiple of Brownian motion starting from some point $c\left(x_0\right)$, which is similar with the conclusion we drew for 2-dimensional case.

Restrictions of GFF on Compact Smooth Hypersurfaces:

As we have seen from above, GFF can be "nicely" restricted to concentric circles ($n=2$) and spheres ($n \geq 3$), and the one-parameter process arising from the restriction is nothing but a Brownian motion. A natural question to ask at this point would be whether GFF can also be restricted to other curves ($n=2$) or hypersurfaces ($n \geq 3$) and what are the laws of the restrictions. Even if the answer to the first question is yes, we can't expect more than Gaussian variables for the second one, because in general, we won't be able to evaluate the variance for restrictions on general curves or hypersurfaces explicitly.

We claim that for general $n \geq 2$, if D is a bounded domain in $\mathbb{R}^n$, and $M \subseteq D$ is a compact smooth hypersurface of $\mathbb{R}^n$, then the GFF indexed by $H\left(D\right)$ can be restricted to M. Here by "compact", we require M to be compact in the topology of $\mathbb{R}^n$. We only sketch the proof for $n=2$, and higher dimension cases will follow from the same idea.

Suppose D is some bounded domain in $\mathbb{R}^2$, and $M=\gamma\left(t\right)$ is smooth curve parametrized by length. Since M has to be bounded, then the length of $\gamma\left(t\right)$ has to be finite and denote the length by T; moreover, the Gaussian curvature of $\gamma\left(t\right)$ is bounded both above and below, which implies that there exists a constant $C\left(\gamma\right)>0$, such that for any $0<t,\,s<T$,

$$\frac{1}{C\left(\gamma\right)}\left|t-s\right| \leq \left|\gamma\left(t\right)-\gamma\left(s\right)\right| \leq C\left(\gamma\right)\left|t-s\right|.$$

In other words, the Euclidean distance between two points on $\gamma\left(t\right)$ is comparable with the distance between them along the curve. To verify that the GFF can be restricted on γ, we need to show that the following integral is finite:

$$\int_0^T\int_0^T \log\left|\gamma\left(t\right)-\gamma\left(s\right)\right|\left|\dot{\gamma}\left(t\right)\right|\left|\dot{\gamma}\left(s\right)\right|dtds.$$

First notice that $\left|\dot{\gamma}\left(t\right)\right|=\left|\dot{\gamma}\left(s\right)\right|=1$, because γ is parametrized by the length. Then, since the domain of integration is bounded, the only problem comes from the situation when t and s get close.

Let's fix $s \in (0, T)$ for a moment, and divide the integral for t into three pieces:$(0,\, s-\delta)$, $(s-\delta,\, s+\delta)$ and $(s+\delta,\, T)$, where $0<\delta<1$ is some constant to be determined. When $t \in (0,\, s-\delta) \cup (s+\delta,\, T)$, t is at least δ away from s, then

$$\left|\int_{s+\delta}^{T} \log\left|\gamma(t)-\gamma(s)\right| dt\right| \vee \left|\int_{0}^{s-\delta} \log\left|\gamma(t)-\gamma(s)\right| dt\right| \\ \leq T\left(C(\gamma)\left(\left|\log(T-s)\right| \vee \left|\log s\right|\right) - \frac{\log\delta}{C(\gamma)}\right).$$

We are allowed to take $\delta = \frac{1}{2}\left(s \wedge (T-s)\right) \wedge 1$, then the inequality above can be rewritten as

$$\left|\int_{s+\delta}^{T} \log\left|\gamma(t)-\gamma(s)\right| dt\right| \vee \left|\int_{0}^{s-\delta} \log\left|\gamma(t)-\gamma(s)\right| dt\right| \\ \leq T\left(C(\gamma) + \frac{1}{C(\gamma)}\right)\left(\left|\log(T-s)\right| \vee \left|\log s\right|\right).$$

For the second piece of the integral, we have

$$\left|\log\left|\gamma(t)-\gamma(s)\right|\right| \leq \left(C(\gamma) + \frac{1}{C(\gamma)}\right)\left|\log\left|t-s\right|\right|,$$

and,

$$\int_{s-\delta}^{s+\delta} \left|\log\left|t-s\right|\right| dt = 2\delta\left(1-\log\delta\right).$$

Therefore,

$$\left|\int_{s-\delta}^{s+\delta} \log\left|\gamma(t)-\gamma(s)\right| dt\right| \leq 2T\left(C(\gamma) + \frac{1}{C(\gamma)}\right)\left(\left|\log(T-s)\right| \vee \left|\log s\right| + 1\right).$$

Moreover, it's easy to see that

$$\int_0^T \left|\log(T-s)\right| ds = \int_0^T \left|\log s\right| ds = \begin{cases} T - T\log T, & 0<T\leq 1; \\ 2 - T + T\log T, & T \geq 1. \end{cases}$$

Thus, we have shown that the GFF indexed by $H(D)$ for $D \subseteq \mathbb{R}^2$ can be restricted on a smooth curve with finite length.

In higher dimensional space, $n \geq 3$, $M \subseteq D$ is a compact smooth hypersurface, then the Gaussian curvature of M will be bounded both above and below. We divide M into finitely many small patches through partition of unity. Then locally on each patch, the Euclidean distance between two points is comparable with the length of the geodesic joining them, which is also the radius of exponential map on that patch. Then we can carry out the similar arguments as $n = 2$ to get the conclusion.

Massive GFF:

Starting from this section, we want to introduce the massive GFF. Generally speaking, the massive GFF is closely related to the GFF because they have similar behaviors locally; but the massive GFF "lives" in the whole space $\mathbb{R}^n$ instead of any bounded domain, and this will allow us to apply Fourier analysis whose advantage will be seen later.

To start, suppose instead of Dirichlet inner product $(\cdot,\,\cdot)_\nabla$ on some bounded domain $D \subseteq \mathbb{R}^n$, we consider the Sobolev H^1 inner product on the whole space. Namely, for any $\phi \in C_c^\infty(\mathbb{R}^n)$, define the inner product as

$$(\phi,\,\phi)_{H^1} = \sqrt{\int_{\mathbb{R}^n} |\phi(x)|^2\,dx + \int_{\mathbb{R}^n} |\nabla\phi|^2\,dx},$$

and denote the closure of $C_c^\infty(\mathbb{R}^n)$ under $(\cdot,\,\cdot)_{H^1}$ as $H^1(\mathbb{R}^n)$. By standard Sobolev theory, we know that $H^1(\mathbb{R}^n)$ is a Hilbert space under $(\cdot,\,\cdot)_{H^1}$, and hence we can consider a GFF indexed by $H^1(\mathbb{R}^n)$. Suppose h is an instance of such a GFF, then $\left\{(h,\,g)_{H^1} : \, g \in H^1(\mathbb{R}^n)\right\}$ is a family of centered Gaussian variables, and the covariance for $g_1,\,g_2 \in H^1(\mathbb{R}^n)$ is given by

$$\mathbb{E}\left[(h,\,g_1)_{H^1}(h,\,g_2)_{H^1}\right] = (g_1,\,g_2)_{H^1}\,.$$

Suppose $\rho(x) = (I-\Delta)\,\phi$, where $I-\Delta$ is the standard Helmholtz operator on $\mathbb{R}^n$. Similarly, if we consider the action of $(h,\,\rho)$ formally in the sense of L^2 inner product, then the covariance is the following:

$$\mathbb{E}\left[(h,\,\rho_1)(h,\,\rho_2)\right] = \left(\rho_1,\,(I-\Delta)^{-1}\rho_2\right). \qquad (1)$$

We can certainly proceed as in the GFF case, namely, finding out the Green function of Helmholtz operator on $\mathbb{R}^n$; however, it turns out that only when n is small, the formula of the fundamental solution to Helmholtz equation is explicit enough. The alternative approach we need here is provided by Fourier analysis. Let's go back to the definition of $(\cdot,\,\cdot)_{H^1}$ for $\phi \in C_c^\infty(\mathbb{R}^n)$. Since ϕ is certain L^2 integrable on $\mathbb{R}^n$, then we can apply Parseval identity, which means the following is true:

$$(\phi,\,\phi)_{H^1}^2 = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)d\xi.$$

If we take $\psi(\xi) = \hat{\phi}(\xi)\left(1+|\xi|^2\right)$, then apparently $\psi \in L^2(\mathbb{R}^n)$; hence we can take the inverse Fourier transform of ψ, and denote it as ρ. ρ is also L^2 integrable, and moreover,

$$\int_{\mathbb{R}^n}\frac{|\hat{\rho}(\xi)|^2}{1+|\xi|^2}d\xi = \int_{\mathbb{R}^n}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)d\xi < \infty.$$

Therefore, if we consider $H^1(D)$ as the closure of $C_c^\infty(\mathbb{R}^n)$ under the inner product defined by Fourier transforms, then the covariance of the action of $(h,\,\rho)$ can also be given by

$$\mathbb{E}\left[(h,\,\rho_1)(h,\,\rho_2)\right]=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\left|\hat{\rho}(\xi)\right|^2}{1+\left|\xi\right|^2}d\xi. \qquad (2)$$

For $n=2,\,3$, we can calculate the fundamental solution to Helmholtz equation on $\mathbb{R}^n$ explicitly, and if we use (1) to define the covariance between $(h,\,\rho_1)$ and $(h,\,\rho_2)$, we have when $n=2$,

$$\mathbb{E}\left[(h,\,\rho_1)(h,\,\rho_2)\right]=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho_1(x)\,\rho_2(y)\,\frac{K_0(|x-y|)}{2\pi}dxdy,$$

where K_0 is the Bessel K function with order 0; when $n=3$,

$$\mathbb{E}\left[(h,\,\rho_1)(h,\,\rho_2)\right]=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho_1(x)\,\rho_2(y)\,\frac{e^{-|x-y|}}{4\pi\,|x-y|}dxdy.$$

By standard knowledge about Bessel function [7], one has

$$K_0(|\xi|)\approx-\log\left(\frac{1}{2}\,|\xi|\right)+\gamma+\mathcal{O}\left(\xi^2\right)$$

when ξ is close to 0 from the right, where γ is some constant not depending on ξ. So it's clear that when x, y are close,

$$\frac{K_0(|x-y|)}{2\pi}\approx-\frac{\log(|x-y|)}{2\pi}+\gamma^{'}+\mathcal{O}\left((x-y)^2\right),$$

and

$$\frac{e^{-|x-y|}}{4\pi\,|x-y|}\approx\frac{1}{4\pi\,|x-y|}+\eta+\mathcal{O}\left((x-y)^2\right),$$

where both $\gamma^{'}$ and η are constants not depending on x or y.

Therefore, if we recall the argument that we used in the previous section to prove that the GFF indexed by $H(D)$ can be restricted to compact smooth curves ($n=2$) and hypersurfaces($n=3$), then it's not hard to see that the same is true for the massive GFF indexed by $H^1(\mathbb{R}^n)$. In other words, when $n=2$, $\mathbb{E}\left[(h,\,\lambda)^2\right]<\infty$, where λ is the uniform norm on any smooth curve $\gamma(t)$ with finite length in $\mathbb{R}^2$, so that h can be restricted on $\gamma(t)$; similarly when $n=3$, h can be restricted on any compact smooth hypersurface in $\mathbb{R}^3$. For general n, we will adopt the definition (2) for the covariance in discussing the restrictions of the massive GFF in the following section.

Restrictions of Massive GFF:

As we have seen from above, for any restriction distribution χ, χ can act on the massive GFF indexed by $H^1(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n}\frac{\left|\hat{\chi}(\xi)\right|^2}{1+\left|\xi\right|^2}d\xi<\infty.$$

It's clear that we have to require the Fourier transform of χ decays fast enough in ξ when $|\xi|$ is large.

Let's see a simple example first. Suppose χ_t is the renormalized restriction on the $n-1$ sphere $S^{n-1}(0,t)$, then the Fourier transform of χ_t is defined as

$$\hat{\chi}_t(\xi) = \frac{1}{t^{n-1}\sigma_n}\int_{S^{n-1}(0,t)} e^{i(x,\xi)} d\sigma(x).$$

Clearly $\hat{\chi}_t(\xi)$ is a radial function in ξ, and we can rewrite the integral above as

$$\begin{aligned}\hat{\chi}_t(|\xi|) &= \frac{1}{\sigma_n}\int_0^{\pi} \sin^{n-2}\theta e^{it|\xi|\cos\theta} d\theta \\ &= \frac{2^{\frac{n-2}{2}}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\sigma_n} J_{\frac{n-2}{2}}(t|\xi|)(t|\xi|)^{\frac{2-n}{2}} \\ &= A_n J_{\frac{n-2}{2}}(t|\xi|)(t|\xi|)^{\frac{2-n}{2}},\end{aligned}$$

where $A_n = 2^{-\frac{n}{2}}\pi^{1-\frac{n}{2}}(n-2)!$ and $J_{\frac{n-2}{2}}(\xi)$ is the Bessel J function with order $\frac{n-2}{2}$. We can compute (2) in polar coordinate as the following:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\hat{\chi}_t(\xi)|^2}{1+|\xi|^2} d\xi &= A_n^2 \int_{\mathbb{R}^n} \frac{\left|J_{\frac{n-2}{2}}(t|\xi|)\right|^2}{\left(1+|\xi|^2\right)(t|\xi|)^{n-2}} d\xi \\ &= A_n^2\sigma_n \int_0^{\infty} r^{n-1} \frac{\left|J_{\frac{n-2}{2}}(tr)\right|^2}{(1+r^2)(tr)^{n-2}} dr \\ &= \frac{A_n^2\sigma_n}{t^{n-2}} \int_0^{\infty} \frac{r\left|J_{\frac{n-2}{2}}(tr)\right|^2}{1+r^2} dr. \qquad (3)\end{aligned}$$

By the standard theory of Bessel J function [7], we know that the leading term in the asymptotic expansion of $J_{\frac{n-2}{2}}(\xi)$ as $|\xi| \to \infty$ is $|\xi|^{-\frac{1}{2}}$. In other words,

$$J_{\frac{n-2}{2}}(\xi) \sim \frac{1}{\sqrt{|\xi|}}$$

as $|\xi| \to \infty$; hence

$$\frac{r\left|J_{\frac{n-2}{2}}(tr)\right|^2}{1+r^2} \sim \frac{1}{1+r^2}$$

as $r \to \infty$. Therefore, the integral in (3) is finite. So we get the conclusion that the massive GFF indexed by $H(\mathbb{R}^n)$ can be restricted on $n-1$ spheres.

In general, for integral (2) to converge, $\hat{\chi}(\xi)$ has to decay fast enough in $|\xi|$ as $|\xi| \to \infty$. To be specific, we have to require ,

$$\hat{\chi}(\xi) \sim |\xi|^{-\frac{n-2}{2}-\epsilon},$$

for some $\epsilon > 0$. In other words, for any hypersurface $M \subseteq \mathbb{R}^n$, if the restriction distribution χ_M on M has a Fourier transform that decays fast enough at infinity, then the massive GFF can be restricted on M. The works of E. Stein and his collaborators provide us a possible family of such $M^{'}$s. Namely, the following has been proved in [5].

Suppose S is a smooth hypersurface in $\mathbb{R}^n$ whose Gaussian curvature is non-zero everywhere and let $d\chi = \psi d\sigma$, where $d\sigma$ is the standard surface measure on S and ψ is some smooth compactly supported function in $\mathbb{R}^n$ whose support intersects S and the intersection is small enough, then

$$\hat{\chi}(\xi) \sim \frac{1}{|\xi|^{\frac{n-1}{2}}}.$$

This implies, if S is smooth with non-zero Gaussian curvature everywhere, then the Fourier transform of the restriction on S locally provides enough decay at infinity. Therefore, with partition of unity, at least we can conclude that:

Suppose S is a smooth compact hypersurface in $\mathbb{R}^n$ with non-zero Gaussian curvature everywhere, then the massive GFF indexed by $H(\mathbb{R}^n)$ can be restricted on S, for any $n \geq 1$.

We summarize as the following: suppose h is an instance of the massive GFF, M is a smooth hypersurface in $\mathbb{R}^n$ and χ is the restriction distribution on M, then if we adopt the definition (1) for the restrictions of h, then, when $n = 2$ or $n = 3$, and M is compact, h can be restricted on M; if we adopt the definition (2), then for any $n \geq 1$, when M is compact with non-zero Gaussian curvature everywhere, h can be restricted on M.

References:

1. L. Evans. Partial Differential Equations. American Mathematical Society. GSM. Vol. 19. 2002.

2. J. Glimm, A. Jaffe. Quantum Physics: A Functional Integral Point of View. Springer- Verlag. 1981.

3. X. Hu, J. Miller, Y. Peres. Thick Points of the Gaussian Free Field. arXiv:0902.3842v1 [math.PR] February 2009.

4. S. Sheffield. Gaussian free fields for mathematicians. Probability Theory and Related Fields. Vol 139. PP 521-541. 2007.

5. E. Stein. Harmonic Anlysis. Princeton University Press. 1995.

6. D. Stroock. Probability Theory: an Analytic View. 2nd edition. In preparation.

7. G. Watson. A Treatise on the Theory of Bessel Functions. Cambridge Univeristy Press. 1944.