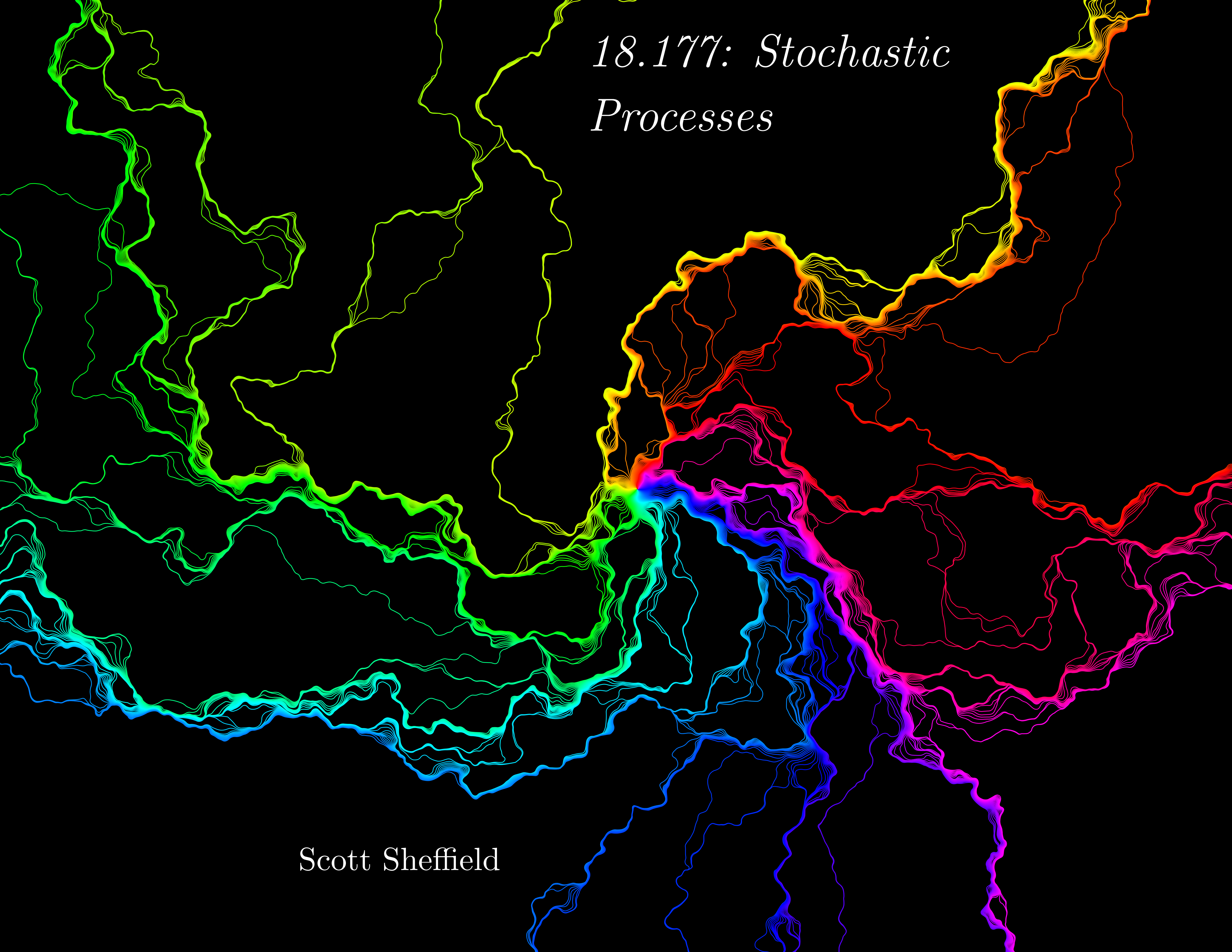
18.177: Stochastic Processes



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Course topics:

Discrete statistical mechanics: percolation, Ising and Potts models, dimers, uniform spanning trees.

Schramm-Loewner evolution (SLE): a family of random fractal curves in the plane with connections to discrete models and to conformal field theory.

Gaussian free field: a two-dimensional-time analog of Brownian motion.

Random planar maps and Liouville quantum gravity: what is the most natural way to define a (uniformly) *random* Riemannian manifold?

The *standard Gaussian* on n-dimensional Hilbert space

has density function $e^{-(v,v)/2}$ (times an appropriate constant). We can write a sample from this distribution as

$$\sum_{i=1}^{n} \alpha_i v_i$$

where the v_i are an orthonormal basis for $\mathbb{R}^n$ under the given inner product, and the α_i are mean zero, unit variance Gaussians.

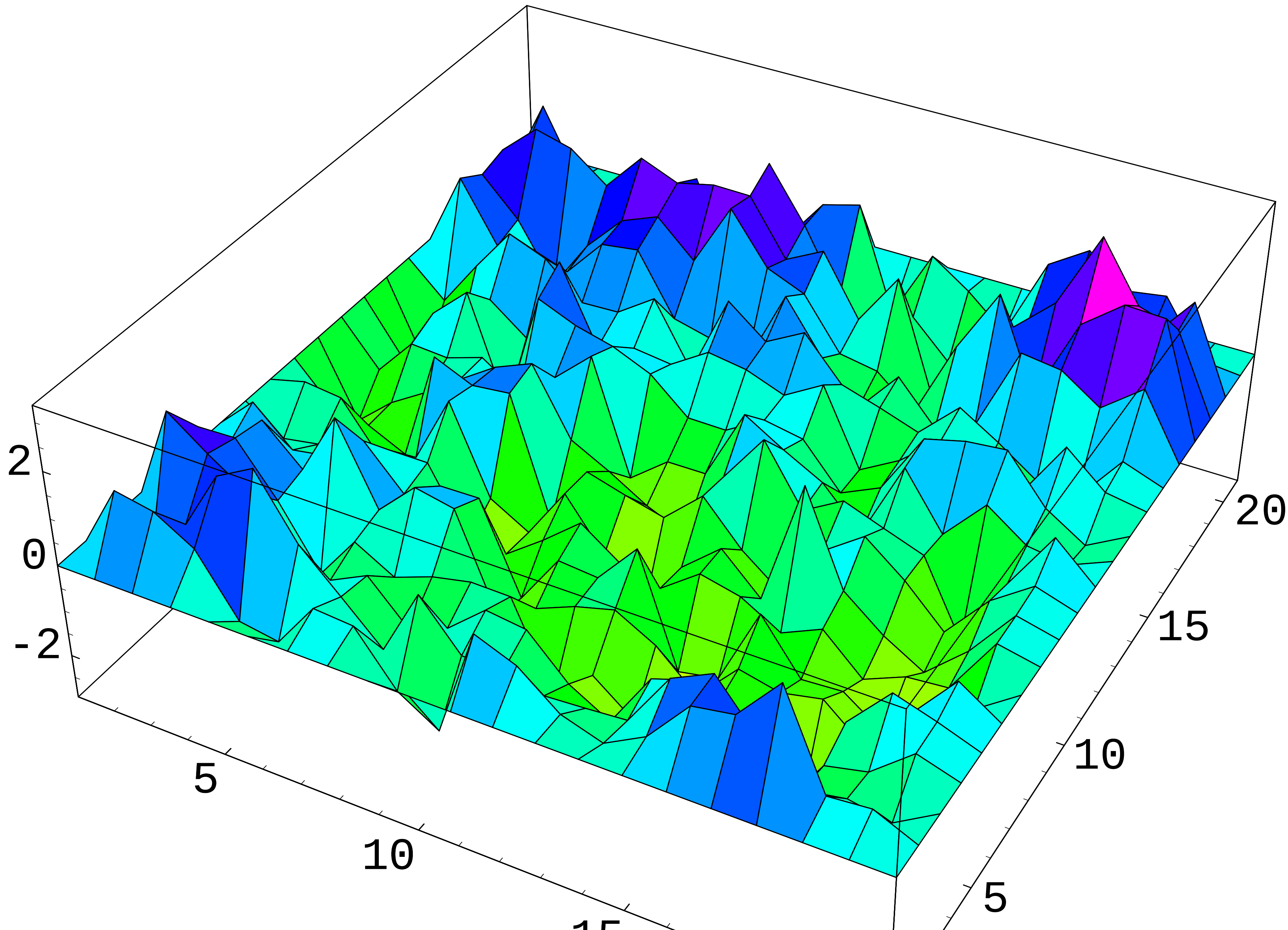
The discrete Gaussian free field

Let f and g be real functions defined on the vertices of a planar graph Λ. The **Dirichlet inner product** of f and g is given by

$$(f,g)_\nabla = \sum_{x \sim y} (f(x) - f(y))\,(g(x) - g(y))\,.$$

The value $H(f) = (f,f)_\nabla$ is called the **Dirichlet energy of f.**
Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



Some DGFF properties:

Zero boundary conditions: The Dirichlet form $(f, f)_\nabla$ is an inner product on the space of functions with zero boundary, and the DGFF is a standard Gaussian on this space.

Other boundary conditions: DGFF with boundary conditions f_0 is the same as DGFF with zero boundary conditions *plus* a deterministic function, which is the (discrete) harmonic interpolation of f_0 to Λ.

Markov property: **Given** the values of f on the boundary of a subgraph Λ' of Λ, the values of f on the remainder of Λ' have the law of a DGFF on Λ', with boundary condition given by the observed values of f on $\partial\Lambda'$.

The continuum Gaussian free field

is a “standard Gaussian” on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx dy.$$

The GFF is the formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—inner products (h, ϕ) are well defined whenever ϕ is sufficiently smooth.