The SLE trace is a continuous path

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Abstract

This is a final project in the class 18.177 - Stochastic Processes in Physics. A proof of the fact that the SLE$_\kappa$ trace is a continuous path will be given in the case $\kappa \neq 8$, following the argument in [RS].

Let's review the definition of chordal SLE$_\kappa$. For $\kappa > 0$ let $\xi(t) := \sqrt{\kappa} B_t$ where B_t is Brownian motion on $\mathbb{R}$ started from $B_0 = 0$. For each $z \in \overline{\mathbb{H}} \setminus \{0\}$ we let $g_t(z)$ be the solution of the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \tag{1}$$

which exists as long as $g_t(z) - \xi(t)$ is bounded away from zero. Denote by $\tau(z)$ the first time τ such that 0 is a limit point of $g_t(z) - \xi(t)$ as $t \nearrow \tau$. Set

$$H_t := \{z \in \mathbb{H} : \tau(z) > t\}, \quad K_t := \{z \in \overline{\mathbb{H}} : \tau(t) \leq t\}.$$

For all $t \geq 0$, H_t is open and K_t is compact. The parameterized collection of maps $(g_t : t \geq 0)$ is called chordal SLE$_\kappa$. For every $t \geq 0$ the map $g_t : H_t \to \mathbb{H}$ is a conformal homeomorphism and H_t is the unbounded component of $\mathbb{H} \setminus K_t$.

Two important basic properties of chordal SLE are summarized in the following proposition.

Proposition 1. 1. *Scaling property: The process $(z,t) \mapsto \tilde{g}_t = \alpha^{-1/2} g_{\alpha t}(\sqrt{\alpha} z)$ has the same law as the process $(z,t) \mapsto g_t(z)$.*

2. *Let $t_0 > 0$. The map $(z,t) \mapsto \hat{g}_t(z) := g_{t+t_0} \circ g_{t_0}^{-1}(z + \xi(t_0)) - \xi(t_0)$ has the same las as the map $(z,t) \mapsto g_t(z)$; moreover, $(\hat{g}_t)_{t \geq 0}$ is independent of $(g(t))_{0 \leq t \leq t_0}$.*

Proof. We have

$$\partial_t \tilde{g}_t(z) = \frac{2}{\tilde{g}_t(z) - \alpha^{-1/2}\xi(\alpha t)}, \quad \partial_t \hat{g}_t(z) = \frac{2}{\hat{g}_t(z) - (\xi(t+t_0) - \xi(t_0))}$$

so the result follows from the scaling property and translation invariance of Brownian motion. The independence claim follows from the Markov property of Brownian motion. □

We will use the following notation

$$f_t := g_t^{-1}, \quad \hat{f}_t(z) = f_t(z + \xi(t))$$

The **trace** γ of SLE is defined by

$$\gamma(t) := \lim_{z \to 0} \hat{f}_t(z)$$

where z tends to 0 within $\mathbb{H}$. If the limit does not exist, let $\gamma(t)$ denote the set of all limit points. We say that the SLE trace is a continuous path if the limit exists for every t and $\gamma(t)$ is a continuous function of t. Our goal is to show precisely this.

The following technical lemma gives a sufficient condition for a local martingale to be a martingale.

Lemma 2. *Let B_t be a standard one dimensional Brownian motion and let a_t be a progressive real valued locally bounded process. Suppose X_t satisfies*

$$X_t = \int_0^t a_s dB_s$$

and that for every $t > 0$ there is a finite constant $c(t)$ such that

$$a_s^2 \leq c(t) X_s^2 + c(t) \tag{2}$$

for all $s \in [0, t]$ a.s. Then X is a martingale.

Proof. We already know that X is a local martingale. Take a large $M > 0$ and let $T := \inf\{t : |X_t| \geq M\}$. Then $Y_t := X_{t \wedge T}$ is a martingale. Put $f(t) := \mathbf{E}[Y_t^2]$. Itô's isometry then gives

$$f(t') = \mathbf{E}\left[\int_0^{t'} a_s^2 1_{s<T} ds\right].$$

Our assumption (2) therefore implies that for $t' \in [0,t]$

$$f(t') \leq c(t)t' + c(t)\int_0^{t'} f(s)ds, \tag{3}$$

since $(1+X_s^2)1_{s<T} \leq 1 + Y_s^2$. If t' is the least $s \in [0,t]$ such that $f(s) \geq e^{2c(t)s}$, we get by (3)

$$\begin{aligned} e^{2c(t)t'} &\leq f(t') \leq c(t)t' + c(t)\int_0^{t'} f(s)ds < c(t)t' + c(t)\int_0^{t'} e^{2c(t)s}ds \\ &= c(t)t' + \tfrac{1}{2}e^{2c(t)t'} - \tfrac{1}{2} \end{aligned}$$

i.e. $e^{2c(t)t'} < 2c(t)t' - 1$, a contradiction. Hence $f(s) < e^{2c(t)s}$ for all $s \in [0,t]$. Thus,

$$\mathbf{E}[\langle X, X\rangle_{t\wedge T}] = \mathbf{E}[\langle Y, Y\rangle_t] = \mathbf{E}[Y_t^2] = f(t) < e^{2c(t)t}.$$

By letting $M \to \infty$ we get by monotone convergence $\mathbf{E}[\langle X, X\rangle_t] \leq e^{2c(t)t} < \infty$. Hence X is a martingale (by [RY99, IV.1.25]).

□

We will need estimates for the moments of $|\hat{f}_t'|$. For convenience, we let B_t be a two-sided Brownian motion. Equation (1) can also be solved for negative t and g_t is a conformal map from $\mathbb{H}$ into a subset of $\mathbb{H}$ when $t < 0$. Note that Proposition 1 also holds in this generalized setting. The following lemma will be useful:

Lemma 3. *For all fixed $t \in \mathbb{R}$ the map $z \mapsto g_{-t}(z)$ has the same distribution as the map $z \mapsto \hat{f}_t(z) - \xi(t)$.*

Proof. Fix $t_0 \in \mathbb{R}$ an let

$$\hat{g}_t(z) := g_{t+t_0} \circ g_{t_0}^{-1}(z + \xi(t_0)) - \xi(t_0).$$

The generalized Proposition 1 gives that $(z,t) \mapsto \hat{g}_t(z)$ has the same distribution as $(z,t) \mapsto g_t(z)$. Hence $z \mapsto g_{-t_0}(z)$ has the same distribution as $z \mapsto \hat{g}_{-t_0}(z) = \hat{f}_{t_0}(z) - \xi(t_0)$.

□

Note that (1) gives

$$\partial_t \mathrm{Im}(g_t(z)) = -\frac{2\mathrm{Im}(g_t(z))}{|g_t(z) - \xi(t)|^2} \tag{4}$$

so $\mathrm{Im}(g_t(z))$ is monotone decreasing in t for every $z \in \mathbb{H}$. For $z \in \mathbb{H}$ and $u \in \mathbb{R}$ set

$$T_u = T_u(z) := \sup\{t \in \mathbb{R} : \mathrm{Im}(g_t(z)) \geq e^u\}.$$

We claim that for all $z \in \mathbb{H}$ a.s. $T_u \neq \pm\infty$. Put $\bar{\xi}(t) := \sup\{|\xi(s)| :\in [0,t]\}$ and note that by (1) we have $\partial_t |g_t(z)| \leq |\partial_t g_t(z)| = 2|g_t(z) - \xi(t)|^{-1} \leq 2(|g_t(z)| - \bar{\xi}(t))^{-1}$ whenever $|g_t(z)| > \bar{\xi}(t)$. This implies $|g_t(z)| \leq |z| + \bar{\xi}(t) + 2\sqrt{t}$ for all $t < \tau(z)$, since if this were not true, we would by continuity have the inverse inequality for all t in an interval $(t_0, t_1]$ and equality at t_0 and thus

$$\begin{aligned} |z| + \bar{\xi}(t_1) + 2\sqrt{t_1} &< |g_{t_1}(z)| = |g_{t_0}(z)| + \int_{t_0}^{t_1} \partial_t |g_t(z)| dt \\ &\leq |g_{t_0}(z)| + \int_{t_0}^{t_1} \frac{2}{|g_t(z)| - \bar{\xi}(t)} dt \\ &< |g_{t_0}(z)| + \int_{t_0}^{t_1} \frac{1}{\sqrt{t}} dt \\ &= |g_{t_0}(z)| + 2\sqrt{t_1} - 2\sqrt{t_0} \\ &= |z| + \bar{\xi}(t_0) + 2\sqrt{t_1} \end{aligned}$$

i.e. $\bar{\xi}(t_1) < \bar{\xi}(t_0)$, a contradiction. From (4) we then get

$$-\partial_t \log \mathrm{Im}(g_t(z)) = \frac{2}{|g_t(z) - \xi(t)|^2} \geq \frac{2}{(|z| + 2\bar{\xi}(t) + 2\sqrt{t})^2}.$$

By the law of iterated logarithms, $\limsup_{t\to\infty} B_t/\sqrt{2t \log\log t} = 1$ a.s. which implies that the right hand side is not integrable over $[0,\infty)$ nor over $(-\infty, 0]$. Hence a.s. we have $\lim_{t\to\pm\infty} \log \mathrm{Im}(g_t(z)) = \mp\infty$ and thus $|T_u| < \infty$.

We will need the formula

$$\begin{aligned} \partial_t \log |g_t'(z)| &= \mathrm{Re}\left(\frac{\partial_z \partial_t g_t(z)}{g_t'(z)}\right) = \mathrm{Re}\left(\frac{1}{g_t'(z)} \partial_z \frac{2}{g_t(z) - \xi(t)}\right) \\ &= -2\mathrm{Re}\left(\frac{1}{(g_t(z) - \xi(t))^2}\right). \end{aligned} \tag{5}$$

Set $u = u(z,t) := \log \mathrm{Im}(g_t(z))$ and remember that by (4) we have

$$\partial_t u = -\frac{2}{|g_t(z) - \xi(t)|^2}. \tag{6}$$

By (5) and (6) we get

$$\partial \log |g_t'(z)| = \frac{\mathrm{Re}((g_t(z) - \xi(t))^2)}{|g_t(z) - \xi(t)|^2}. \tag{7}$$

Now, fix some $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$. For every $u \in \mathbb{R}$, let

$$z(u) := g_{T_u(\hat{z})}(\hat{z}) - \xi(T_u), \quad x(u) := \mathrm{Re}(z(u))$$
$$y(u) := \mathrm{Im}(z(u)) = e^u, \quad \psi(u) := \frac{\hat{y}}{y(u)}|g'_{T_u}(\hat{z})|.$$

Theorem 4. *Let $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$ as above. Assume $\hat{y} \neq 1$ and set $\nu := -sign(\log \hat{y})$. Let $b \in \mathbb{R}$ and define a and λ by*

$$a := 2b + \nu\kappa b(1-b)/2, \quad \lambda := 4b + \nu\kappa b(1-2b)/2. \tag{8}$$

Set

$$F(\hat{z}) = F_b(\hat{z}) := \hat{y}^a \mathbf{E}\left[(1 + x(0)^2)^b |g_{T_0(\hat{z})}(\hat{z})|^a\right].$$

Then

$$F(\hat{z}) = \left(1 + (\hat{x}/\hat{y})^2\right)^b \hat{y}^\lambda.$$

Proof. Note that by (6) we have

$$du = -2|z|^{-2}dt$$

Put

$$\hat{B}(u) := -\sqrt{2/\kappa}\int_{t=0}^{T_u} |z|^{-1} d\xi.$$

Then $\hat{B}$ is a Brownian motion w.r.t. $\int_0^t 2|z|^{-2}ds = -u$ and hence also w.r.t. u. Set $M_u := \psi(u)\hat{F}(z(u))$, where $\hat{F}(x+iy) = \left(1 + (x/y)^2\right)^b y^\lambda$. Itô's formula gives

$$dM_u = -2M\frac{bx}{x^2+y^2}d\xi = \sqrt{2\kappa}M\frac{bx}{\sqrt{x^2+y^2}}d\hat{B}.$$

Hence M is a local martingale and Lemma 2 tells us that M is a martingale. Thus we have

$$\hat{F}(\hat{z}) = \psi(\hat{u})^a \hat{F}(\hat{z}) = \mathbf{E}[\psi(0)^a \hat{F}(z(0))] = \hat{y}^a \mathbf{E}\left[(1 + x(0)^2)^b |g_{T_0(\hat{z})}(\hat{z})|^a\right].$$

□

With the aid of Theorem 4 we can get the following estimates for $|\hat{f}'_t|$:

Corollary 5. *Let $b \in [0, 1 + \frac{4}{\kappa}]$, and define λ and a as in (8) with $\nu = 1$. There is a constant $C(\kappa, b)$, depending only on κ and b, such that the following estimate holds for all $t \in [0,1], y, \delta \in (0,1]$ and $x \in \mathbb{R}$:*

$$\mathbf{P}\left[|\hat{f}'_t(x+iy)| \geq \delta y^{-1}\right] \leq C(\kappa, b)(1 + x^2/y^2)^b (y/\delta)^\lambda \theta(\delta, a - \lambda), \tag{9}$$

where

$$\theta(\delta, s) = \begin{cases} \delta^{-s} & \text{if } s > 0, \\ 1 + |\log \delta| & \text{if } s = 0, \\ 1 & \text{if } s < 0. \end{cases}$$

Proof. Note that the condition on b is equivalent to $a \geq 0$. If we make sure $C(\kappa, b) \geq 1$ then the right hand side is at least 1 when $\delta \leq y$ so we may assume that $\delta > y$. Take $z = x + iy$. By Lemma 3, $\hat{f}_t'(z)$ has the same distribution as $g_{-t}'(z)$. We put $u_1 := \log \mathrm{Im}(g_{-t}(x + iy))$ and observe that

$$\frac{|g_{-t}'(z)|}{|g_{T_u(z)}'(z)|} \leq e^{|u - u_1|},$$

since $|\partial_u \log |g_t'(z)|| \leq 1$ by (7). Because $t, y \leq 1$, there is a constant $c \geq 1$ such that $u_1 \leq c$. Therefore,

$$\mathbf{P}\left[|g_{-t}'(z)| \geq e^c \delta y^{-1}\right] \leq \sum_{j=\lceil \log y \rceil}^{0} \mathbf{P}\left[|g_{T_j(z)}'(z)| \geq \delta y^{-1}\right],$$

since $\log y \leq u_1 \leq c$ implies there is an integer j between $\lceil \log y \rceil$ and 0 such that $|j - u_1| \leq c$. By the Schwarz lemma, $y|g'(z)| \leq \mathrm{Im}(g(z))$ if $g : \mathbb{H} \to \mathbb{H}$ is holomorphic, so the above gives

$$\mathbf{P}\left[|g_{-t}'(z)| \geq e^c \delta y^{-1}\right] \leq \sum_{j=\lceil \log \delta \rceil}^{0} \mathbf{P}\left[|g_{T_j(z)}'(z)| \geq \delta y^{-1}\right]. \tag{10}$$

By scale invariance, $g_{T_j(z)}'(z)$ has the same distribution as $g_{T_0(e^{-j}z)}'(e^{-j}z)$. Hence

$$\mathbf{E}\left[y^a e^{-ja} |g_{T_j(z)}'(z)|^a\right] = y^a e^{-ja} \mathbf{E}\left[|g_{T_0(e^{-j}z)}'(e^{-j}z)|^a\right] \leq F_b(e^{-j}z),$$

where F_b is as in Theorem 4. Thus we get

$$\begin{aligned} \mathbf{P}\left[|g_{T_j(z)}'(z)| \geq \delta y^{-1}\right] &= \mathbf{P}\left[|g_{T_j(z)}'(z)|^a y^a \delta^{-a} \geq 1\right] \\ &\leq \mathbf{E}\left[|g_{T_j(z)}'(z)|^a y^a \delta^{-a}\right] \\ &\leq \delta^{-a} e^{ja} F_b(e^{-j}z). \end{aligned}$$

Since $j \geq \log \delta > \log y$ the imaginary part of $e^{-j}z$ remains below 1 so we get by Theorem 4

$$F_b(e^{-j}z) = (1 + x^2/y^2)^b e^{-j\lambda} y^\lambda.$$

Consequently, by (10)

$$\mathbf{P}\left[|g_{-t}'(z)| \geq e^c \delta y^{-1}\right] \leq (1 + x^2/y^2)^b \delta^{-a} y^\lambda \sum_{j=\lceil \log \delta \rceil}^{0} e^{j(a-\lambda)}.$$

If $a = \lambda$, the sum is bounded by $1 + |\log \delta| = \theta(\delta, 0)$. If $a > \lambda$, the sum is bounded by the constant $(1 - e^{\lambda - a})^{-1}$, which only depends on κ and b, and $\theta(\delta, a - \lambda) = \delta^{\lambda - a}$. If $a < \lambda$ the sum is bounded by $(1 - e^{a-\lambda})^{-1}\delta^{a-\lambda}$ and $\theta(\delta, a - \lambda) = 1$. Therefore, if we put $\hat{C}(\kappa, b) = 1 + |a - \lambda|(1 - e^{-|a-\lambda|})^{-1}$ we have for all choices of δ, y

$$\mathbf{P}\left[|\hat{f}_t'(z)| \geq e^c \delta y^{-1}\right] \leq \hat{C}(\kappa, b)(1 + x^2/y^2)^b (y/\delta)^\lambda \theta(\delta, a - \lambda).$$

Put $\delta' = e^{-c}\delta$ to get (9) with $C(\kappa, b) = \hat{C}(\kappa, b)(e^{ca} + e^{c\lambda}(c + 1))$.

□

The following theorem shows that $\hat{f}_t(0) = f_t(\xi(t))$ exists as a radial limit and is continuous.

Theorem 6. *Define*

$$H(y, t) := \hat{f}_t(iy), \quad y > 0, t \in [0, \infty).$$

If $\kappa \neq 8$, then a.s. $H(y, t)$ extends continuously to $[0, \infty) \times [0, \infty)$.

Proof. Fix $\kappa \neq 8$. By scale invariance, it suffices to show continuity of H on $[0, \infty) \times [0, 1)$. For $j, k \in \mathbb{N}$, with $k < 2^{2j}$ we define the rectangle

$$R(j, k) := [2^{-j-1}, 2^{-j}] \times [k2^{-2j}, (k + 1)2^{-2j}],$$

and put

$$d(j, k) := \text{diam } H(R(j, k)).$$

We take $b = (8 + \kappa)/(4\kappa) < 1 + 4/\kappa$ and let a and λ be given by (8) with $\nu = 1$. Then $\lambda = 2 + (\kappa - 8)^2/(16\kappa) > 2$ so we can pick σ such that $0 < \sigma < (\lambda - 2)/\max\{a, \lambda\}$. To begin with, we want to show that

$$\sum_{j=0}^{\infty} \sum_{k=0}^{2^{2j}-1} \mathbf{P}[d(j, k) \geq 2^{-j\sigma}] < \infty. \tag{11}$$

Fix a pair (j, k). Set $t_0 = (k + 1)2^{-2j}$ and inductively,

$$t_{n+1} := \sup\{t < t_n : |\xi(t) - \xi(t_n)| = 2^{-j}\}.$$

Let N be the least $n \in \mathbb{N}$ such that $t_n \leq t_0 - 2^{-2j}$, set $t_\infty := t_0 - 2^{-2j} = k2^{-2j}$ and

$$\hat{t}_n := \max\{t_n, t_\infty\}.$$

The scaling property of Brownian motion shows that there is a constant $\rho < 1$, independent of j and k, such that $\mathbf{P}[N > 1] = \rho$ and the Markov property gives $\mathbf{P}[N \geq m+1 \mid N \geq m] \leq \rho$. Thus, $\mathbf{P}[N > m] = \rho^m$.

For every $s \geq 0$, the map $\hat{f}_s$ is measurable with respect to the σ-algebra generated by $\xi(t)$ for $t \in [0, s]$ while $\hat{t}_n$ is determined by $\xi(t)$ for $t \geq t_n$. The strong Markov property then gives for every $n \in \mathbb{N}, s \in [t_\infty, t_0]$ and $\delta > 0$

$$\mathbf{P}\left[|\hat{f}'_{\hat{t}_n}(i2^{-j})| > \delta \mid \hat{t}_n = s\right] = \mathbf{P}\left[|\hat{f}'_s(i2^{-j}) > \delta\right]$$

which yields

$$\begin{aligned} \mathbf{P}\left[|\hat{f}'_{\hat{t}_n}(i2^{-j})| > \delta \mid \hat{t}_n > t_\infty\right] &= \mathbf{E}\left[\mathbf{P}\left[|\hat{f}'_{\hat{t}_n}(i2^{-j})| > \delta \mid \hat{t}_n\right] \mid \hat{t}_n > t_\infty\right] \\ &\leq \sup_{s \in (0,1]} \mathbf{P}\left[|\hat{f}'_s(i2^{-j})| > \delta\right]. \end{aligned}$$

Therefore, we get the estimate

$$\begin{aligned} & \mathbf{P}\left[\exists n \in \mathbb{N} : |\hat{f}'_{\hat{t}_n}(i2^{-j}) > \delta\right] \\ \leq\ & \mathbf{P}\left[|\hat{f}'_{\hat{t}_\infty}(i2^{-j})| > \delta\right] + \sum_{n=0}^{\infty} \mathbf{P}[\hat{t}_n > t_\infty] \mathbf{P}\left[|\hat{f}'_{\hat{t}_n}(i2^{-j})| > \delta \mid \hat{t}_n > t_\infty\right] \\ \leq\ & (1 + \mathbf{E}[N]) \sup_{s \in (0,1]} \mathbf{P}\left[|\hat{f}'_s(i2^{-j})| > \delta\right] \\ \leq\ & O(1) \sup_{s \in (0,1]} \mathbf{P}\left[|\hat{f}'_s(i2^{-j})| > \delta\right]. \end{aligned} \tag{12}$$

Note that $a - \lambda = (\kappa^2 - 64)/(32\kappa)$. If $\kappa > 8$, then $a > \lambda$ and Corollary 5 gives

$$\sup_{s \in (0,1]} \mathbf{P}\left[|\hat{f}'_s(i2^{-j})| > 2^j 2^{-j\sigma}/j^2\right] \leq O(1) 2^{-j(\lambda - a\sigma)} j^{2a} \leq O(1) 2^{-j(2+\epsilon)}, \tag{13}$$

for some $\epsilon = \epsilon(\kappa) > 0$, since $\sigma < (\lambda - 2)/a$. If $\kappa < 8$, then $a < \lambda$ and Corollary 5 gives

$$\sup_{s \in (0,1]} \mathbf{P}\left[|\hat{f}'_s(i2^{-j})| > 2^j 2^{-j\sigma}/j^2\right] \leq O(1) 2^{-j(\lambda - \lambda\sigma)} j^{2\lambda} \leq O(1) 2^{-j(2+\epsilon)}, \tag{14}$$

for some $\epsilon = \epsilon(\kappa) > 0$, since $\sigma < (\lambda - 2)/\lambda$. Now let S be the rectangle

$$S := \{x + iy : |x| \leq 2^{-j+3}, y \in [2^{-j-1}, 2^{-j+3}]\}.$$

We want to show that

$$H(R(j,k)) \subset \bigcup_{n=0}^{N} \hat{f}_{\hat{t}_n}(S) \tag{15}$$

and

$$\hat{f}_{\hat{t}_n}(S) \cap \hat{f}_{\hat{t}_{n+1}}(S) \neq \emptyset \quad \forall n \in \mathbb{N}. \tag{16}$$

Let $t \in [\hat{t}_{n+1}, \hat{t}_n]$ and $y \in [2^{-j-1}, 2^{-j}]$. Then we can write

$$H(y,t) = \hat{f}_t(iy) = \hat{f}_{\hat{t}_{n+1}}\left(g_{\hat{t}_{n+1}}(\hat{f}_t(iy)) - \xi(\hat{t}_{n+1})\right).$$

We will prove (15) by showing that $g_{\hat{t}_{n+1}}(\hat{f}_t(iy)) - \xi(\hat{t}_{n+1}) \in S$. Define $\phi(s) = g_s(\hat{f}_t(iy))$ for $s \leq t$. Then $\phi(t) = iy + \xi(t)$ and by (1)

$$\partial_s \phi(s) = 2(\phi(s) - \xi(s))^{-1}.$$

Note that $\partial_s \mathrm{Im}(\phi(s)) < 0$ and hence $\mathrm{Im}(\phi(s)) \geq \mathrm{Im}(\phi(t)) \geq 2^{-j-1}$. This gives $|\partial_s \phi(s)| \leq 2^{j+2}$ and since $|t - \hat{t}_{n+1}| \leq 2^{-2j}$ we then get $|\phi(\hat{t}_{n+1}) - \phi(t)| \leq 2^{2-j}$. Since $|\xi(t) - \xi(\hat{t}_{n+1}| \leq 2^{1-j}$ we get

$$g_{\hat{t}_{n+1}}(\hat{f}_t(iy)) - \xi(\hat{t}_{n+1})| = \phi(\hat{t}_{n+1}) - \phi(t) + iy + \xi(t) - \xi(\hat{t}_{n+1}) \in S$$

which gives $\hat{f}_t(iy) \in \hat{f}_{\hat{t}_{n+1}}(S)$ and verifies (15). If we take $t = \hat{t}_n$ in the above we get $\hat{f}_{\hat{t}_n}(iy) \in \hat{f}_{\hat{t}_{n+1}}(S)$ which verifies (16). By the Koebe distortion theorem ([Pom92, 1.3]) $|\hat{f}_t'(z)|/|\hat{f}_t'(i2^{-j})|$ is bounded by some constant (independent of j and t) if $z \in S$ and thus we have

$$\mathrm{diam}(\hat{f}_t(S)) \leq O(1)2^{-j}|\hat{f}_t'(i2^{-j})|.$$

Therefore, we get from (15) and (16)

$$\begin{aligned} d(j,k) &\leq \sum_{n=0}^{N} \mathrm{diam}(\hat{f}_{\hat{t}_n}(S)) \leq O(1)2^{-j}\sum_{n=0}^{N}|\hat{f}_{\hat{t}_n}'(i2^{-j})| \\ &\leq O(1)2^{-j}N\max\{|\hat{f}_{\hat{t}_n}'(i2^{-j})| : n = 0,1,\ldots,N\}. \end{aligned} \tag{17}$$

By (12), (13) and (14) we get

$$\begin{aligned} \mathbf{P}[d(j,k) > 2^{-j\sigma}] &\leq \mathbf{P}\left[O(1)2^{-j}N\max\{|\hat{f}_{\hat{t}_n}'(i2^{-j})| : n = 0,1,\ldots,N\} > 2^{-j\sigma}\right] \\ &\leq \mathbf{P}[O(1)N > j^2] + \mathbf{P}\left[\max\{|\hat{f}_{\hat{t}_n}'(i2^{-j})| : n = 0,1,\ldots,N\} > 2^j 2^{-j\sigma}/j^2\right] \\ &\leq \rho^{j^2/O(1)} + O(1)2^{-j(2+\epsilon)} \leq O(1)2^{-j(2+\epsilon)} \end{aligned}$$

which proves (11).

A consequence of (11) is that a.s. there are at most finitely many pairs $j, k \in \mathbb{N}$ with $k \leq 2^{2j} - 1$ such that $d(j,k) > 2^{-j\sigma}$. Thus we have $d(j,k) \leq C(\omega)2^{-j\sigma}$ for all j, k, where the constant $C(\omega)$ is random. Let (y', t') and (y'', t'') be points in $(0,1)^2$. Let j_1 be the largest integer less than $\min\{-\log_2 y', -\log_2 y'', -\frac{1}{2}|t' - t''|\}$. Then $y', y'' < 2^{-j_1}$ and $|t' - t''| < 2^{-2j_1}$ so we get the estimate

$$|H(y', t') - H(y''.t'')| \leq \sum_{j=j_1}^{\infty} (d(j, k'_j) + d(j, k''_j)) \leq O(1)C(\omega)2^{-\sigma j_1},$$

where $R(j, k'_j)$ is a rectangle meeting the line $t = t'$ and $R(j, k''_j)$ is a rectangle meeting the line $t = t''$. This shows that for every $t_0 \in [0,1)$ the limit of $H(y,t)$ as $(y,t) \to (0, t_0)$ exists and thereby extends the definition of H to a continuous function on $[0, \infty) \times [0, 1)$.

□

It follows from [LSW] that the theorem holds also when $\kappa = 8$.

Now we get a criterion for hulls to be generated by a continuous path.

Theorem 7. *Let $\xi : [0, \infty) \to \mathbb{R}$ be continuous and let g_t be the corresponding solution to (1). Assume that $\beta(t) := \lim_{y \searrow 0} g_t^{-1}(\xi(t) + iy)$ exists and is continuous for all $t \in [0, \infty)$. Then g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ and H_t is the unbounded connected component of $\mathbb{H} \setminus \beta([0,t])$ for every $t \in [0, \infty)$.*

In the proof, we will need the following basic properties of conformal maps. Suppose $g : \Omega \to \mathbb{H}$ is a conformal homeomorphism. If $\alpha : [0,1) \to \Omega$ is a path such that the limit $l_1 = \lim_{t \nearrow 1} \alpha(t)$ exits, then $l_2 = \lim_{t \nearrow 1} g(\alpha(t))$ exists too. (It is important that $\mathbb{H}$ is a nice domain.) Moreover, $\lim_{t \nearrow 1} g^{-1}(tl_2)$ exists and equals l_1. Therefore, if $\tilde{\alpha} : [0,1) \to \Omega$ is another path such that $\lim_{t \nearrow 1} \tilde{\alpha}(t)$ exists and $\lim_{t \nearrow 1} g(\alpha(t)) = \lim_{t \nearrow 1} g(\tilde{\alpha}(t))$, then $\lim_{t \nearrow 1} \alpha(t) = \lim_{t \nearrow 1} \tilde{\alpha}(t)$. A proof of these statements can be found in [Pom92, Proposition 2.14] and [Ahl73, Theorem 3.5].

Proof. Let $S(t) \subset \overline{\mathbb{H}}$ be the set of limit points of $g_t^{-1}(t)$ as $t \to \xi(t)$ in $\mathbb{H}$. Fix $t_0 \geq 0$ and assume $z_0 \in S(t_0)$. We will show that $z_0 \in \overline{\beta([0, t_0))}$ and hence $z_0 \in \beta([0, t_0])$. Fix some $\epsilon > 0$. Put

$$t' := \sup\{t \in [0, t_0] : K_t \cap \overline{D(z_0, \epsilon)} = \emptyset\},$$

where $D(z_0, \epsilon)$ is the open disk of radius ϵ about z_0. To begin with, we show that

$$\beta(t') \in \overline{D(z_0, \epsilon)}. \tag{18}$$

Since $z_0 \in S(t_0)$, $D(z_0, \epsilon) \cap H_{t_0} \neq \emptyset$. Take $p \in D(z_0, \epsilon) \cap H_{t_0}$ and let $p' \in K_{t'} \cap \overline{D(z_0, \epsilon)}$ (this set is nonempty by the definition of t' and the fact that $z_0 \in K_{t_0}$). Let p'' be the first point of the line segment from p to p' which is in $K_{t'}$. We will show that $\beta(t') = p''$. Let L be the line segment $[p, p'')$ and note that $L \subset H_{t'}$. Then $g_{t'}(L)$ is a curve in $\mathbb{H}$ terminating at a point $x \in \mathbb{R}$. If $x \neq \xi(t')$, then $g_t(L)$ terminates at points $x(t) \neq \xi(t)$ for all $t < t'$ sufficiently close to t'. Because $g_\tau(p'')$ has to hit the singularity $\xi(\tau)$ at some time $\tau \leq t'$, this implies $p'' \in K_t$ for some $t < t'$. But this contradicts the definition of t' and hence shows that $x = \xi(t')$. Now $\beta(t') = p''$ follows because the conformal map g_t^{-1} of $\mathbb{H}$ cannot have to different limits along two arcs with the same terminal point.

Now we have established (18) and since $\epsilon > 0$ was arbitrary, we conclude that $z_0 \in \overline{\beta([0, t_0))}$ and hence $z_0 \in \beta([0, t_0])$. This gives $S(t) \subset \beta([0, t])$ for all $t \geq 0$. Now we argue that H_t is the unbounded component of $\mathbb{H} \setminus \overline{\bigcup_{\tau \leq t} S(\tau)}$. First, H_t is connected and disjoint from $\overline{\bigcup_{\tau \leq t} S(\tau)}$. On the other hand, as the argument in the previous paragraph shows, $\partial H_t \cap \mathbb{H}$ is contained in $\overline{\bigcup_{\tau \leq t} S(\tau)}$. Therefore, H_t is the unbounded connected component of $\mathbb{H} \setminus \overline{\bigcup_{\tau \leq t} S(\tau)} = \mathbb{H} \setminus \beta([0, 1])$. Since β is a continuous path, it follows from [Pom92, Theorem 2.1] that g_t^{-1} extends continuously to $\overline{\mathbb{H}}$, which also proves that $S(t) = \{\beta(t)\}$.

□

Now we have all the results needed to prove:

Theorem 8. *The following statement holds almost surely. For every $t \geq 0$ the limit*

$$\gamma(t) := \lim_{z \to 0, z \in \mathbb{H}} \hat{f}_t(z)$$

exists, $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ is a continuous path and H_t is the unbounded component of $\mathbb{H} \setminus \gamma([0, t])$.

Proof. By Theorem 6, a.s. $\lim_{y \searrow 0} \hat{f}_t(iy)$ exists for all t and is continuous. Therefore we can apply Theorem 7 and the result follows.

□

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