18.177 Final Project: Thick points of a Gaussian Free Field

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Disclaimer

What follows is a paper based entirely on the paper "Thick Points of the Gaussian Free Field" by Xiaoyu Hu, Jason Miller, and Yuval Peres [1]. The purpose of this paper is both to convey that I read and understood the paper [1], and also to hopefully serve as a more intuitive/less rigorous explanation of the statements and proofs of Theorems 1.1 and 1.2 in [1]. After reading this paper, the reader is encouraged to read the paper [1] to fill in the rigorous details. For the most part I have kept the notation the same as in [1] which should ease the transition.

Abstract

Thick points of a Gaussian Free Field (GFF) are points z such that the average value of z on the disk $D(z,r)$ centered at z with radius r grows at a certain rate as $r \to 0$. Specifically, z is called an a-thick point if $\lim_{r\to 0} \frac{\int_{D(z,r)} F(x)\ dx}{\pi r^2 \log(\frac{1}{r})} = \sqrt{\frac{a}{\pi}}$. We will show that the Hausdorff dimension of the set of a-thick points of a GFF over a two dimensional domain U is equal to $2-a$ almost surely for $0 \le a \le 2$.

1 Introduction

Given a discrete graph and the Discrete Gaussian Free Field (DGFF) on that graph, it makes sense to talk about "the set of points with height greater than or equal to a" for each instance of the DGFF. The same statement, however, makes no sense when you are discussing a GFF over a continuous domain U as for the random field F only exists as a distribution and so the expression $F(x)$ for a fixed point x in U is meaningless. We can however evaluate the "average value of F over the set A" for certain sets A. In particular, the expression $\int_A F(x)\ dx$ can be rigorously interpreted as the pairing $(F, 1_A)$ when the function $1_A(x)$ is a member of $H_0^1(U)$. For this paper we will only need the fact that $\int_A F(x)\ dx$ is well defined when A is a disk, the boundary of a disk, or a square (For a proof of these facts see chapter 2 of [1]). So while we cannot talk about the "set of points above height a", we can talk about the "set of points whose average over smaller and smaller disks has an asymptotic limit depending on a". With this in mind, we define an *a-thick point* to be a point z in U such that

$$\lim_{r\to 0} \frac{\int_{D(z,r)} F(x)\ dx}{\pi r^2 \log(\frac{1}{r})} = \sqrt{\frac{a}{\pi}}$$

and we define an *a-circle thick point* to be a point z in U such that

$$\lim_{r\to 0} \frac{\int_{\partial D(z,r)} F(x)\ dx}{2\pi r \log(\frac{1}{r})} = \sqrt{\frac{a}{\pi}}$$

(We will see later why the extra factor $\log(\frac{1}{r})$ appears in the denominator). We use the notation $T(a;U)$ and $T^C(a;U)$ to denote the set of a-thick points and a-circle thick points respectively. Our main theorem will be the following:

Theorem 1.1.

$$\dim_H(T(a;U)) = \dim_H(T^C(a;U)) = 2 - a, \ a.s.$$

We will prove this theorem in three steps:

1. We introduce the set

$$T^{C,s}_{\geq}(a;U) = \{z : \limsup_{r\to 0} \frac{\int_{\partial D(z,r)} F(x)\,dx}{2\pi r \log(\frac{1}{r})} \geq \sqrt{\frac{a}{\pi}}\}$$

(we just introduced it!) and show that

$$T^C(a;U) \leq T(a;U) \leq T^{C,s}_{\geq}(a;U)$$

2. We show that

$$\dim_H(T^{C,s}_{\geq}(a;U)) \leq 2 - a, \ a.s.$$

3. We show that

$$\dim_H(T^C(a;U)) \geq 2 - a, \ a.s.$$

It is clear that our main theorem will be proven once we complete these three steps.

2 Preliminary Facts

Before proceeding to these three steps, we will need some preliminary facts. First, we define

$$F(z,r) = \frac{\int_{\partial D(z,r)} F(x)\,dx}{2\pi r}$$

to be the *circle average process*. This quantity appears in both the definitions of $T^C(a;U)$ and $T^{C,s}_{\geq}(a;U)$ so it will clearly be useful. It is called a "process" because as $r \to 0$, $F(z,r)$ behaves like a Brownian motion run at time $t = \log(\frac{1}{r})$. We show this formally below:

Theorem 2.1. *Let* $B(z,t) = \sqrt{2\pi}F(z,e^{-t})$*. Then* $B(z,t) - B(z,t_1)$ *has the law of standard Brownian motion for* $t \geq t_1$*.*

Remark 2.2. This is equivalent to saying $F(z,r)$ is equal in law to $\frac{1}{\sqrt{2\pi}}B(z,\log(\frac{1}{r}))$ where $B(z,t)$ is a standard Brownian motion (that might not start at 0).

Proof. We first note that $F(z,r) = (F,\eta(z,r))$ where $\eta(z,r)$ is the uniform probability measure on $\partial D(z,r)$. Therefore, by standard facts about GFF, we have that

$$\begin{aligned}
Cov(B(z,t),B(z,s)) &= 2\pi Cov(F(z,e^{-t}),F(z,e^{-s})) \\
&= 2\pi Cov((F,\eta(z,e^{-t})),(F,\eta(z,e^{-s}))) \\
&= 2\pi \int_U \int_U \eta(z,e^{-t})(x)\eta(z,e^{-s})(y)G(x,y)\,dxdy
\end{aligned}$$

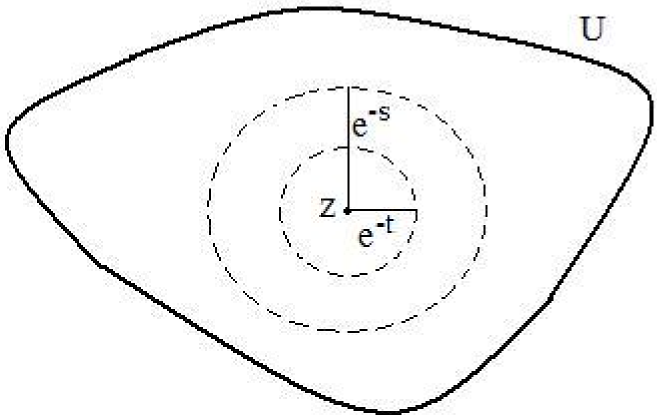
where $G(x,y) = \frac{-1}{2\pi}(\log|x-y| - \text{harmonic extension})$ is the standard Green's function. Intuitively, the quantity

$$\int_U \int_U \eta(z,e^{-t})(x)\eta(z,e^{-s})(y)G(x,y)\,dxdy$$

is what you get if you solve the Poisson partial differential equation

$$\begin{cases} \Delta u = \eta(z, e^{-s}) \\ u = 0 \text{ on } \partial U \end{cases}$$

and then take the average value of u on the circle $\partial D(z, e^{-t})$. Suppose $s \leq t$ then our picture looks like this:



We know that the solution to

$$\begin{cases} \Delta u = \delta_z \\ u = 0 \text{ on } \partial U \end{cases}$$

is given by $u = \frac{-1}{2\pi} \log|x - z| - HE_z(x)$, where $HE_z(x)$ is the harmonic extension of the boundary values of $\frac{-1}{2\pi} \log|x - z|$ on ∂U. Therefore, by symmetry, the solution to our original PDE is given by this but truncated so that u is constant in the disk $D(z, e^{-s})$. That is, we have that

$$u = \frac{-1}{2\pi} \log(\max(|x - z|, e^{-s})) - HE_z(x)$$

And so the average of this on the $\partial D(z, e^{-t})$ is easily computed as the average of $\frac{-1}{2\pi} \log \max(|x - z|, e^{-s})$ is equal to $\frac{-1}{2\pi} \log(e^{-s}) = \frac{s}{2\pi}$ and the average of $HE_z(x)$ over $D(z, e^{-t})$ is simply $HE_z(z)$ since $HE_z(x)$ is harmonic. In particular, $HE_z(z)$ only depends on z and not t. And so we have that for $s \leq t$,

$$\begin{aligned} Cov(B(z,t), B(z,s)) &= 2\pi \int_U \int_U \eta(z, e^{-t})(x) \eta(z, e^{-s})(y) G(x,y) \, dxdy \\ &= s + 2\pi HE_z(z) \end{aligned}$$

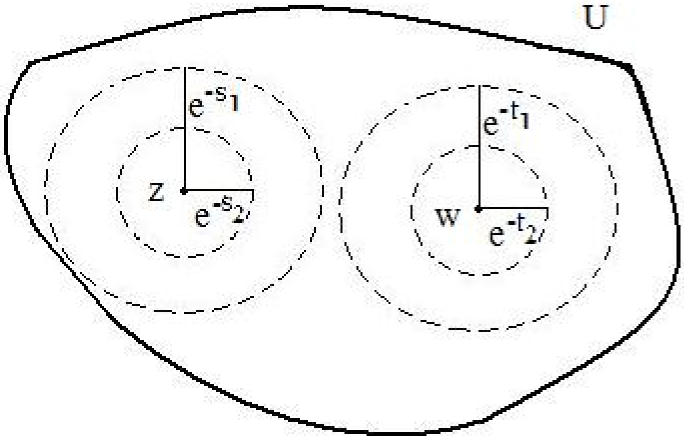
It therefore follows that for $t_1 \leq s \leq t$,

$$Cov(B(z,t) - B(z,t_1), B(z,s) - B(z,t_1)) = (s - t_1)$$

and so we have shown that $B(z,t) - B(z,t_1)$ has the law of standard Brownian motion started from 0. $\square$

So for a fixed z, $F(z,r)$ looks like a time-scaled Brownian motion as $r \to 0$. Of course, $F(z,r)$ and $F(w,s)$ are correlated for each pair of points (z,w). However, by the Markov property of GFF, once the rings $\partial D(z,r)$ and $\partial D(w,s)$ stop overlapping, the increments of the respective processes will be independent. We state this more precisely as the following theorem:

Theorem 2.3. *Given z, w and $s_1 \leq s \leq s_2$, $t_1 \leq t \leq t_2$ such that the annuli $D(z, e^{-s_1})\backslash D(z, e^{-s_2})$ and $D(z, e^{-t_1})\backslash D(z, e^{-t_2})$ are disjoint, then the Brownian motions $B(z, s) - B(z, s_1)$ for $s_1 \leq s \leq s_2$ and $B(z, t) - B(z, t_1)$ for $t_1 \leq t \leq t_2$ are independent.*



As $F(z, r)$ behaves like a Brownian motion we have that it is continuous in r for fixed z. In fact, we have a stronger modulus of continuity (after taking a modification of F if necessary) in both r and z. This is given in the following theorem whose proof we omit (but can be found in [1])

Theorem 2.4. *For every $0 < \gamma < \frac{1}{2}$ and $\varepsilon, \delta > 0$, there exists an $M = M(\gamma, \varepsilon, \zeta)$ such that*

$$|F(z, r) - F(w, s)| \leq M(\log \frac{1}{r})^{\zeta} \frac{|(z, r) - (w, s)|^{\gamma}}{r^{\gamma(1+\varepsilon)}}$$

for $r, s \in (0, 1]$ with $\frac{1}{2} \leq \frac{r}{s} \leq 2$.

3 The Three Steps

We are now ready to complete the three steps of our proof.

Step 1 (Show that $T^C(a; U) \leq T(a; U) \leq T^{C,s}_{\geq}(a; U)$)

First we use our circle average process $F(z, r)$ and its corresponding Brownian motion process $B(z, t)$ to produce three equivalent definitions for $T^C(a; U)$ and $T^{C,s}_{\geq}(a; U)$:

$$T^C(a; U) = \{z : \lim_{r \to 0} \frac{\int_{\partial D(z,r)} F(x)\, dx}{2\pi r \log(\frac{1}{r})} = \sqrt{\frac{a}{\pi}}\} = \{z : \lim_{r \to 0} \frac{\sqrt{\pi} F(z, r)}{\log(\frac{1}{r})} = \sqrt{a}\} = \{z : \lim_{t \to \infty} \frac{B(z, t)}{\sqrt{2}t} = \sqrt{a}\}$$

$$T^{C,s}_{\geq}(a; U) = \{z : \limsup_{r \to 0} \frac{\int_{\partial D(z,r)} F(x)\, dx}{2\pi r \log(\frac{1}{r})} \geq \sqrt{\frac{a}{\pi}}\} = \{z : \limsup_{r \to 0} \frac{\sqrt{\pi} F(z, r)}{\log(\frac{1}{r})} \geq \sqrt{a}\} = \{z : \limsup_{t \to \infty} \frac{B(z, t)}{\sqrt{2}t} \geq \sqrt{a}\}$$

These equivalent definitions will be of use in what follows(It is now more intuitively clear why we divide by $\log(\frac{1}{r})$ or t as without dividing, there is no hope of achieving a limit since a Brownian motion fluctuates wildly). Since $F(z, r)$ is continuous,we have that

$$\int_{D(z,r)} F(x)\, dx = \int_0^r 2\pi s F(z, s)\, ds$$

and so we have as equivalent definitions for $T(a;U)$,

$$T(a;U)=\{z:\lim_{r\to 0}\frac{\int_{D(z,r)}F(x)\,dx}{\pi r^2\log(\frac{1}{r})}=\sqrt{\frac{a}{\pi}}\}=\{z:\lim_{r\to 0}\frac{\int_0^r 2\pi sF(z,s)\,ds}{\pi r^2\log(\frac{1}{r})}=\sqrt{\frac{a}{\pi}}\}$$

By comparing the definitions involving $F(z,r)$, it is clear that

$$T^C(a;U)\leq T(a;U)\leq T^{C,s}_{\geq}(a;U)$$

Step 2 (Show that $\dim_H(T^{C,s}_{\geq}(a;U))\leq 2-a,\ a.s.$)

We show this by showing that for each $\alpha>2-a$, there exists a covering of $T^{C,s}_{\geq}(a;U)$ by balls such that the α-Hausdorff dimension of this covering is arbitrarily small.

Recall the equivalent definitions of $T^{C,s}_{\geq}(a;U)$ given in Step 1. We first show that, by the modulus of continuity, for each $\varepsilon>0$ and $K=\varepsilon^{-1}$, it suffices to look at the values of $F(z,r)$ on the sequence $r_n=n^{-K}$ (or if we are thinking in terms of the Brownian process $B(z,t)$ we are looking at the values at $t_n=\log\frac{1}{r_n}=K\log(n)$). That is, equivalent definitions for $T^{C,s}_{\geq}(a;U)$ are given by

$$T^{C,s}_{\geq}(a;U)=\{z:\limsup_{n\to\infty}\frac{\sqrt{\pi}F(z,r_n)}{\log(\frac{1}{r_n})}\geq\sqrt{a}\}=\{z:\limsup_{n\to\infty}\frac{B(z,t_n)}{\sqrt{2}t_n}\geq\sqrt{a}\}$$

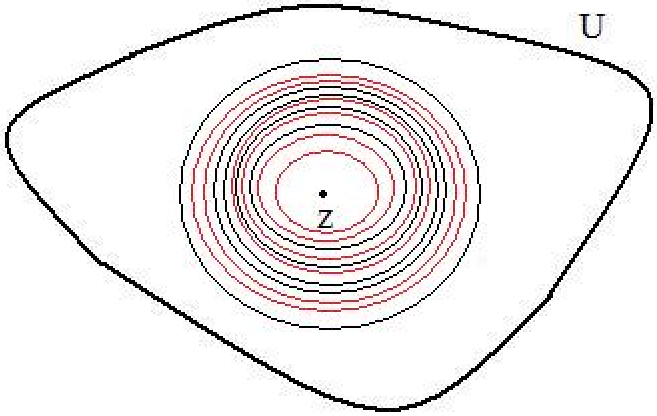
This is true because if we take $\gamma\in(0,1)$, $\zeta\in(0,1)$, and $\varepsilon>0$, by our modulus of continuity we have that for $t_n\leq t\leq t_{n+1}$,

$$\begin{aligned}|B(z,t)-B(z,t_n)| &= \sqrt{2\pi}|F(z,r)-F(z,r_n)|\\ &\leq M(\log\frac{1}{r_n})^{\zeta}\frac{|e^{-t}-r_n|^{\gamma}}{r_n^{\gamma(1+\varepsilon)}}\\ &\leq MK^{\zeta}(\log(n))^{\zeta}\frac{|r_{n+1}-r_n|^{\gamma}}{r_n^{\gamma(1+\varepsilon)}}\end{aligned}$$

Now $|r_n-r_{n+1}|^{\gamma}=|n^{-K}-(n+1)^{-K}|^{\gamma}=O(n^{-(K+1)\gamma})$ and $r_n^{\gamma(1+\varepsilon)}=(n^{-K(1+\varepsilon)})=n^{-K\gamma-\gamma}$. And so

$$|B(z,t)-B(z,t_n)|=O((\log(n))^{\zeta}n^{-(K+1)\gamma}n^{-K\gamma-\gamma})=O((\log(n))^{\zeta})$$

Note that this bound is uniform in n and z since the constant M depended only on γ,ε, and ζ. From this we see that it suffices to look at times t_n or radii r_n since the other times/radii will be controlled by these as we are dividing by t or $\log\frac{1}{r}$ when taking our limit, and both of these quantities are order $\log(n)$. So we now know that whether or not z is in $T^{C,s}_{\geq}(a;U)$ depends only on the average value over a countable number of rings around z. Pictorially, suppose that we color the ring $\partial D(z,r_n)$ red if $\frac{\sqrt{\pi}F(z,r_n)}{\log(\frac{1}{r_n})}\geq\sqrt{a}$ and black otherwise. Then we have that z is in $T^{C,s}_{\geq}(a;U)$ if there are red rings arbitrarily close to it (see picture below):



By a similar argument via the modulus of continuity, we have the following continuity statement for $F(z, r)$ in the z-coordinate:

$$|F(z, r_n) - F(w, r_n)| \leq O((\log(n))^{\zeta}), \text{ if } |z - w| \leq r_n$$

from this modulus of continuity, we see that for other points inside a red ring of radius r_n at z, those points' own rings of radius r_n will be nearly red if not red. More precisely, let $\delta(n) = C(\log(n))^{\zeta-1}$. Then if $\frac{\sqrt{\pi}F(z,r_n)}{\log(\frac{1}{r_n})} \geq \sqrt{a}$ (r_n-th ring is red), then for every w such that $|w - z| < r_n$, we have that $\frac{\sqrt{\pi}F(w,r_n)}{\log(\frac{1}{r_n})} \geq \sqrt{a} - \delta(n)$($r_n$-th ring is nearly red) for a sufficiently large choice of C.

We now have the ingredients necessary to construct a set of balls which covers $T^{C,s}_{\geq}(a; U)$. For each n, we take a set N_n of points $\{z_{nj}\}$ in U such that the union of disks centered at these points with radii $r_n^{1+\varepsilon}$ covers U, and such that there are $O(\frac{1}{r_n^{2(1+\varepsilon)}})$ of these points as $n \to \infty$. We call N_n the *nth net of points* and intuitively it consists of a set of points spaced about a distance $2r_n^{1+\varepsilon}$ apart and such that any point in U is within a distance $r_n^{1+\varepsilon}$ of one of these points.

Assign to z_{nj} (the jth point in the nth net) the event

$$I_n(z_{nj}) = \{\frac{\sqrt{\pi}F(z, r_n)}{\log(\frac{1}{r_n})} \geq \sqrt{a} - \delta(n)\}$$

i.e. this is the event that the r_nth ring surrounding z_{nj} is "nearly red". It is then clear that

$$T^{C,s}_{\geq}(a; U) \subseteq \bigcup_{n \geq N} \{D(z_{nj}, r_n) : I_n(z_{nj}) \text{ is true}\} =: I(a, N)$$

for each N. So for each N, $I(a, N)$ is a random union of balls which covers $T^{C,s}_{\geq}(a; U)$! For $\alpha > 2 - a$, we must show that for each w,

$$H_\alpha(I(a, N)) \to 0 \text{ as } N \to \infty$$

where $H_\alpha(\cdot)$ denotes the α-Hausdorff dimension. Since $I(a, N)$ is a decreasing sequence of sets as $N \to \infty$ and since $H_\alpha(I(a, N)) \geq 0$ it will suffice to show that

$$E[H_\alpha(I(a, N))] \to 0 \text{ as } N \to \infty$$

To this end, we compute that

$$\begin{aligned} E[H_\alpha(I(a,N))] &\leq E[\sum_{n\geq N}\sum_{\{j:I_n(z_{nj})\}}(\text{diam}D(z_{nj},r_n^{1+\varepsilon}))^\alpha] \\ &= \sum_{n\geq N} E[|\{j:I_n(z_{nj})\}|]2^\alpha r_n^{\alpha(1+\varepsilon)} \end{aligned}$$

To compute $E[|\{j:I_n(z_{nj})\}|]$, we note that for a fixed z_{nj}, we have that

$$P(I_n(z_{nj})) = P(\frac{\sqrt{\pi}F(z_{nj},r_n)}{\log(\frac{1}{r_n})} \geq \sqrt{a}-\delta(n)) = O(r_n^{a-o(1)})$$

where here we have used the fact that if $Z\sim N(0,1)$, then $P(|Z|>\lambda)\sim\sqrt{\frac{2}{\pi}}\lambda^{-1}e^{-\frac{\lambda^2}{2}}$ as $\lambda\to\infty$. And so we have that

$$E[|\{j:I_n(z_{nj})\}|] \leq O(\frac{r_n^{a-o(1)}}{r_n^{2(1+\varepsilon)}}) = O(r_n^{a-o(1)-2(1+\varepsilon)})$$

And so finally we have that

$$E[H_\alpha(I(a,N))] \leq \sum_{n\geq N} O(r_n^{a-o(1)-2-\varepsilon+\alpha(1+\varepsilon)})$$

and so taking $\alpha = 2-a+\frac{2+a}{1+\varepsilon}\varepsilon$, we get that

$$E[H_\alpha(I(a,N))] \leq \sum_{n\geq N} O(r_n^{2\varepsilon-o(1))}) = \sum_{n\geq N} O(n^{-2+o(1))}) \to 0 \text{ as } N\to\infty$$

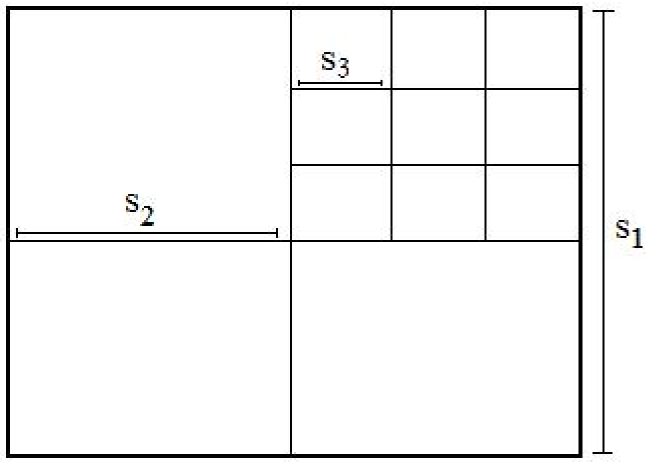
Since α's of this form cover all of the numbers less than $2-a$, Step 2 is proved.

Step 3 (Show that $\dim_H(T^C(a;U)) \geq 2-a,\ a.s.$)

To prove a lower bound on Hausdorff measure, we will need to rely on the Frostman Lemma. This Lemma states that given a set A, if we can find a probability measure μ supported on A (i.e. $\mu(A)=1$) such that its *α-th energy*

$$I_\alpha(\mu) := \int_A\int_A \frac{d\mu(z_1)d\mu(z_2)}{|z_1-z_2|^\alpha}$$

is finite, then $\dim_H(A)\geq\alpha$. So for our purposes, it suffices to find, for each ω, a positive, finite measure (which we can then normalize into a probability measure) with support on $T^C(a;U)$ and whose αth-energy is finite for $\alpha<2-a$. We will construct this measure as the limit of simpler measures. Let H be a square in U and by scaling/translation w.l.o.g. we may take $H=[0,1]^2$. We now divide H into smaller squares



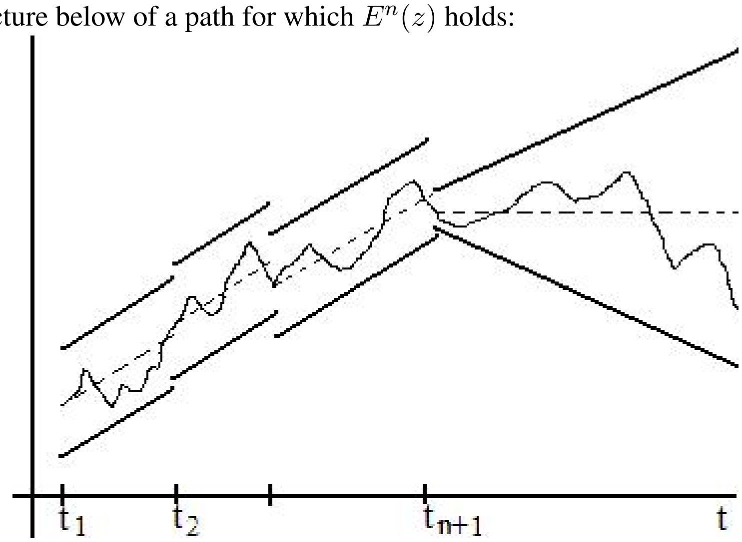
Let $s_n = \frac{1}{n!}$ be the side lengths of the squares and let $t_n = \log \frac{1}{s_n} = \log(n!))$ be the corresponding time values. For each n, we divide H up into s_n^{-2} squares of side length s_n (see the picture above) and let C_n be the set of their centers which we denote by z_{nj}. We will examine the circle average process $F(z,r)$ at each of these centers. For any $z \in U$, we define

$$E_m(z) = \{|B(z,t) - B(z,t_m) - \sqrt{2a}(t-t_m)| \leq \sqrt{t_{m+1} - t_m} \text{ for all } t_m \leq t \leq t_{m+1}\}$$

and

$$F_m(z) = \{|B(z,t) - B(z,t_m)|(t-t_m) + 1 \text{ for all } t_m \leq t \leq t_{m+1}\}$$

and let $E^n(z) = \bigcap_{m \leq n} E_m(z) \cap F_{n+1}(z)$ and call z an n-*perfect thick point* if $E^n(z)$ holds true. We see that if z is an n-perfect thick point, then $B(z,t)$ stays within envelopes which are recalibrated at each t_m up until t_{n+1}. Notice the difference in the form of the envelopes from the E_m events and the F_{n+1} event in the picture below of a path for which $E^n(z)$ holds:



Note that on $E^n(z)$, for $t_m \leq t \leq t_{m+1}$, we have that

$$|B(z,t) - B(z,t_1) - \sqrt{2a}(t-t_1)| \leq \sum_{k=1}^{n} \sqrt{\log(k+1)} = o(m \log m) = o(t) \text{ as } t \to \infty$$

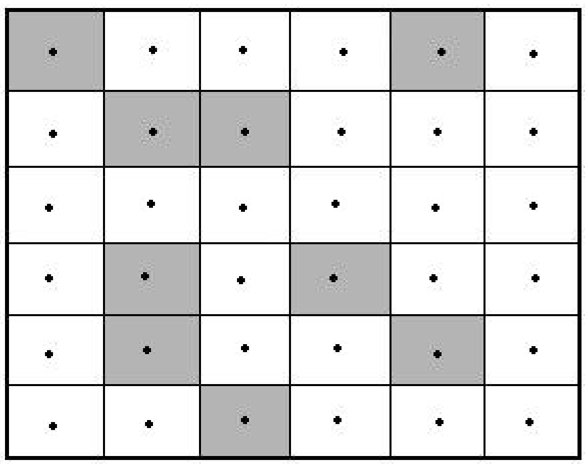
and if $t \geq t_{m+1}$ then

$$|B(z,t) - B(z,t_1)| = O(t)$$

We are now ready to define our sequence of measures. For each n, we define the measure τ_n by

$$\tau_n(A) = \sum_{i=1}^{|C_n|} \frac{1}{P(E^n(z_{ni}))} 1_{\{E^n(z_{ni})\}} |A \cap S(z_{ni}, s_n)|$$

where $S(z_{ni}, s_n)$ is the square centered at z_{ni} with side length s_n, and $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}^2$. Thus, τ_n is a random measure, and intuitively it is generated by looking at the nth set of centers, C_n, determining which of those centers are n-perfect, and then weighting those squares by the reciprocal of the probability of being n-perfect (see picture below).



Assuming that a limit of these measures exists, it will have support on $\limsup_{n\to\infty}(\text{supp}(\tau_n))$(i.e. the set of points which are in supp(τ_n) infinitely often). From the "O(t) bounds" on $B(z,t)$ derived above, it follows that

$$\limsup_{n\to\infty}(\text{supp}(\tau_n)) \subseteq T^C(a;U)$$

so in particular, any limit of the measures τ_n will have support on $T^C(a;U)$, as desired. So it remains to show that we can in fact take a limit of the measures τ_n and that their limiting measure will be finite, positive, and have finite α-energy for $\alpha < 2-a$. For this, we will need that the measures τ_n lie on a compact space and we will show this by exhibiting uniform bounds on $E[\tau_n(H)^2]$ and $E[I_\alpha(\tau_n(H))]$. First we note that clearly for each n,

$$E[\tau_n(H)] = 1$$

We next compute that

$$E[\tau_n(H)^2] = s_n^4 \sum_{i,j=1}^{|C_n|} \frac{1}{P(E^n(z_{ni}))P(E^n(z_{nj}))} P(E^n(z_{ni}) \cap E^n(z_{nj}))$$

and so to find a bound on $E[\tau_n(H)^2]$, we will need to know something about the correlation between the events $E^n(z_{ni})$ and $E^n(z_{nj})$. For this we have the following lemma:

Lemma 3.1. *For $z, w \in H$, let ℓ be such that $w \in S(z, s_\ell) \backslash S(z, s_{\ell+1})$. Then for every $n \geq \ell$ and $\varepsilon > 0$, we have that*

$$P(E^n(z) \cap E^n(w)) \leq O(s_\ell^{-a-\varepsilon}) P(E^n(z)) P(E^n(w))$$

uniformly in z, w, ℓ, n.

Proof. It is a well known result that is $B(t)$ is a Brownian motion, $\mu > 0$, and $T \geq 1$, we have that

$$P(|B(t) - \mu t| \leq \sqrt{T} \text{ for all } 0 \leq t \leq T) \geq Ce^{-\mu\sqrt{T} - \frac{\mu^2 T}{2}}$$

We therefore have that

$$P(E_m(z)) \geq \frac{C}{m^a} \exp(-\sqrt{2a\log(m)})$$

and similarly

$$P(E_m(w)) \geq \frac{C}{m^a} \exp(-\sqrt{2a\log(m)})$$

since the event $E_m(z)$ only depends on the square annulus $D(z, s_m)\backslash D(z, s_{m+1})$, and since $w \in S(z, s_\ell)\backslash S(z, s_{\ell+1})$, we have that for $\ell + 1 < i \leq n$, $1 \leq j \leq n$, $j \neq \ell - 1, \ell, \ell + 1$ the events $E_i(z)$ and $E_j(w)$ are independent.
It therefore follows that

$$P(\bigcap_{1 \leq i \leq \ell+1} E_i(z))P(\bigcap_{\ell-1 \leq j \leq \ell+1} E_j(w)) \geq C^\ell s_\ell^a \exp(-O(\ell\sqrt{\log \ell})) \geq C^\ell S_\ell^{a+\varepsilon}$$

and so finally we have that

$$\begin{aligned}
P(E^n(z) \cap E^n(w)) &= P(\bigcap_{i \leq n} E_i(z) \cap F_{n+1}(z) \cap \bigcap_{j \leq n} E_j(w) \cap F_{n+1}(w)) \\
&= P(\bigcap_{\ell+1 < i \leq n} E_i(z))P(\bigcap_{j \neq \ell-1,\ell,\ell+1} E_j(w))P(\bigcap_{1 \leq i \leq \ell+1} E_i(z)F_{n+1}(z) \cap \bigcap_{j=\ell-1,\ell,\ell+1} E_j(w) \cap F_{n+1}(w)) \\
&\leq P(\bigcap_{\ell+1 < i \leq n} E_i(z))P(\bigcap_{j \neq \ell-1,\ell,\ell+1} E_j(w)) \\
&\leq P(\bigcap_{\ell+1 < i \leq n} E_i(z))P(\bigcap_{j \neq \ell-1,\ell,\ell+1} E_j(w))C^{-\ell} s_\ell^{-a-\varepsilon} P(\bigcap_{1 \leq i \leq \ell+1} E_i(z))P(\bigcap_{\ell-1 \leq j \leq \ell+1} E_j(w)) \\
&= C^{-\ell} s_\ell^{-a-\varepsilon} P(\bigcap_{1 \leq i \leq n} E_i(z))P(\bigcap_{1 \leq j \leq n} E_j(w))
\end{aligned}$$

since $F_{n+1}(z)$ is independent of $E_i(z)$ for $1 \leq i \leq n$ and since $P(F_{n+1}(z)) \geq c > 0$ uniformly in n, z we have that

$$\begin{aligned}
P(E^n(z) \cap E^n(w)) &\leq C^{-\ell} s_\ell^{-a-\varepsilon} P(\bigcap_{1 \leq i \leq n} E_i(z))P(\bigcap_{1 \leq j \leq n} E_j(w)) \\
&\leq O(s_\ell^{-a-\varepsilon})P(\bigcap_{1 \leq i \leq n} E_i(z))P(\bigcap_{1 \leq j \leq n} E_j(w))P(F_{n+1}(z))P(F_{n+1}(w)) \\
&= O(s_\ell^{-a-\varepsilon})P(E^n(z))P(E^n(w))
\end{aligned}$$

□

So with this lemma in hand, we can now finish our computation:

$$\begin{aligned}
E[\tau_n(H)^2] &= s_n^4 \sum_{i,j=1}^{|C_n|} \frac{1}{P(E^n(z_{ni}))P(E^n(z_{nj}))} P(E^n(z_{ni}) \cap E^n(z_{nj})) \\
&\leq s_n^4 \sum_{i=1}^{|C_n|} \sum_{\ell=1}^{n} \frac{s_\ell^2}{s_n^2} \frac{1}{P(E^n(z_{ni}))P(E^n(z_{nj}))} O(s_\ell^{-a-\varepsilon})P(E^n(z_{ni}))P(E^n(z_{nj})) \\
&= \sum_{\ell=1}^{n} s_\ell^2 O(s_\ell^{-a-\varepsilon}) \\
&\leq \sum_{\ell \geq 1} O(s_\ell^{2-a-\varepsilon}) < \infty
\end{aligned}$$

So $E[\tau_n(H)^2]$ is bounded independently of n. Finally, we compute

$$\begin{aligned} E[I_\alpha(\tau_n)] &= \sum_{i,j=1}^{|C_n|} \frac{1}{P(E^n(z_{ni}))P(E^n(z_{nj}))} \int_{S(z_{ni},s_n)} \int_{S(z_{nj},s_n)} \frac{dz_1 dz_2}{|z_1 - z_2|^\alpha} \\ &\leq \sum_{\ell \geq 1} O(s_\ell^{2-a-\varepsilon} s_{\ell+1}^{-\alpha}) \end{aligned}$$

and so $E[I_\alpha(\tau_n)] < \infty$ independently of n for $\alpha < 2 - a$. To recap, we have that

- $E[\tau_n(H)] = 1$ for all n
- $E[\tau_n(H)^2]$ is bounded uniformly in n
- $E[I_\alpha(\tau_n)]$ is bounded uniformly in n

Therefore, there exists a $b, d > 0$ such that

$$P(b \leq \tau_n(H) \leq b^{-1}, I_\alpha(\tau_n) < d) \geq \varepsilon > 0, \text{ for all } n$$

Let $\mathcal{M}_\alpha(b, d)$ be the set of measures μ such that $b \leq \mu(H) \leq b^{-1}$ and $I_\alpha(\mu) < d$. It is known that $\mathcal{M}_\alpha(b, d)$ is compact in the topology of weak convergence. We have shown that for each n,

$$P(\tau_n \in \mathcal{M}_\alpha(b, d)) \geq \varepsilon > 0$$

and so

$$P(\tau_n \in \mathcal{M}_\alpha(b, d) \text{ infinitely often}) \geq \varepsilon > 0$$

For each of the ω on which this event occurs, we can take a subsequence $\tau_{n_k} \in \mathcal{M}_\alpha(b, d)$ which then has a limit point τ in $M_\alpha(b, d)$. This measure will satisfy the Frostman lemma, and so we have shown that

$$P(\dim_H(T^C(a; U)) \geq \alpha) > 0$$

for $\alpha < 2 - a$. Finally, by the Hewitt Savage 0-1 Law, we have that (actually I am not entirely sure how it is applied here)

$$P(\dim_H(T^C(a; U)) \geq \alpha) = 1$$

for $\alpha < 2 - a$. And so we have shown that $\dim_H(T^C(a; U)) \geq 2 - a,\ a.s.$ and thus we have completed Step 3 and so the proof of the main theorem!

References

[1] http://arxiv.org/PS_cache/arxiv/pdf/0902/0902.3842v1.pdf