Defining expectation

Functions of random variables

Motivation
Defining expectation

Functions of random variables

Motivation
Recall: a random variable $X$ is a function from the state space to the real numbers.

Say $X$ is a discrete random variable if (with probability one) it takes one of a countable set of values. For each $a$ in this countable set, write $p(a) := P\{X = a\}$. Call $p$ the probability mass function.

The expectation of $X$, written $E[X]$, is defined by $E[X] = \sum x: p(x) > 0 xp(x)$. Represents weighted average of possible values $X$ can take, each value being weighted by its probability.
Expectation of a discrete random variable

- Recall: a random variable $X$ is a function from the state space to the real numbers.
- Can interpret $X$ as a quantity whose value depends on the outcome of an experiment.
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Say $X$ is a **discrete** random variable if (with probability one) it takes one of a countable set of values.

For each $a$ in this countable set, write $p(a) := P\{X = a\}$. Call $p$ the **probability mass function**.

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$$E[X] = \sum_{x} x p(x),$$

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Simple examples

Suppose that a random variable $X$ satisfies $P\{X = 1\} = .5$, $P\{X = 2\} = .25$ and $P\{X = 3\} = .25$. What is $E[X]$?

Answer:

$E[X] = .5 \times 1 + .25 \times 2 + .25 \times 3 = 1.75$. 

Suppose $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$. Then what is $E[X]$?

Answer:

$E[X] = p$. 

Roll a standard six-sided die. What is the expectation of the number that comes up?

Answer:

$E[X] = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = 3.5$. 

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If the state space $S$ is countable, we can give \textbf{SUM OVER STATE SPACE} definition of expectation:

$$E[X] = \sum_{s \in S} P\{s\} X(s).$$

Example: toss two coins. If $X$ is the number of heads, what is $E[X]$?

State space is \{$(H, H)$, $(H, T)$, $(T, H)$, $(T, T)$\} and summing over state space gives $E[X] = \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 0 = 1$. 
If the state space $S$ is countable, we can give **SUM OVER STATE SPACE** definition of expectation:

$$E[X] = \sum_{s \in S} P\{s\} X(s).$$

Compare this to the **SUM OVER POSSIBLE $X$ VALUES** definition we gave earlier:

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If the state space $S$ is countable, is it possible that the sum $E[X] = \sum_{s \in S} P(\{s\})X(s)$ somehow depends on the order in which $s \in S$ are enumerated?

In principle, yes... We only say expectation is defined when $\sum_{s \in S} P(\{s\})|X(s)| < \infty$, in which case it turns out that the sum does not depend on the order.
A technical point

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If $X$ is a random variable and $g$ is a function from the real numbers to the real numbers then $g(X)$ is also a random variable.
Expectation of a function of a random variable

- If $X$ is a random variable and $g$ is a function from the real numbers to the real numbers then $g(X)$ is also a random variable.
- How can we compute $E[g(X)]$?

$$E[g(X)] = \sum_{s \in S} P(\{s\}) g(X(s))$$

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Suppose that constants $a$, $b$, $\mu$ are given and that $E[X] = \mu$.
- What is $E[X + b]$?
- How about $E[aX]$?
- Generally, $E[aX + b] = aE[X] + b = a\mu + b$. 

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- This implies $E[X] = E[n - X]$. Applying $E[aX + b] = aE[X] + b$ formula (with $a = -1$ and $b = n$), we obtain $E[X] = n - E[X]$ and conclude that $E[X] = n/2$. 
Additivity of expectation

If $X$ and $Y$ are distinct random variables, then can one say that $E[X + Y] = E[X] + E[Y]$?
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Yes. In fact, for real constants $a$ and $b$, we have $E[aX + bY] = aE[X] + bE[Y]$. 

This is called the linearity of expectation.
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- Another way to state this fact: given sample space $S$ and probability measure $P$, the expectation $E[\cdot]$ is a **linear** real-valued function on the space of random variables.
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Another way to state this fact: given sample space $S$ and probability measure $P$, the expectation $E[\cdot]$ is a linear real-valued function on the space of random variables.

Can extend to more variables $E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$. 

Additivity of expectation
Now can we compute expected number of people who get own hats in $n$ hat shuffle problem?

Let $X_i$ be 1 if $i$th person gets own hat and zero otherwise.

What is $E[X_i]$, for $i \in \{1, 2, \ldots, n\}$?

Answer: $1/n$.

Can write total number with own hat as $X = X_1 + X_2 + \ldots + X_n$.

Linearity of expectation gives $E[X] = E[X_1] + E[X_2] + \ldots + E[X_n] = n \times 1/n = 1$. 
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Why should we care about expectation?

- **Laws of large numbers**: choose lots of independent random variables with same probability distribution as $X$ — their average tends to be close to $E[X]$.

- **Economic theory of decision making**: Under "rationality" assumptions, each of us has utility function and tries to optimize its expectation.

- **Financial contract pricing**: under "no arbitrage/interest" assumption, price of derivative equals its expected value in so-called risk neutral probability.

- Comes up everywhere probability is applied.
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- Example: roll $N = 10^6$ dice, let $Y$ be the sum of the numbers that come up. Then $Y/N$ is probably close to 3.5.
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Expected utility when outcome only depends on wealth

- Contract one: I’ll toss 10 coins, and if they all come up heads (probability about one in a thousand), I’ll give you 20 billion dollars.
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Contract two: I’ll just give you ten million dollars.

What are expectations of the two contracts? Which would you prefer?

Can you find a function $u(x)$ such that given two random wealth variables $W_1$ and $W_2$, you prefer $W_1$ whenever $E[u(W_1)] < E[u(W_2)]$?

Let's assume $u(0) = 0$ and $u(1) = 1$. Then $u(x) = y$ means that you are indifferent between getting 1 dollar no matter what and getting $x$ dollars with probability $1/y$. 
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