18.600: Lecture 5

Problems with all outcomes equally like, including a famous hat problem

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Outline

Equal likelihood

A few problems

Hat problem

A few more problems
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A few more problems
Equal likelihood

- If a sample space $S$ has $n$ elements, and all of them are equally likely, the each one has to have probability $1/n$. 

\[
\text{Answer: } \frac{|A|}{|S|}, \text{ where } |A| \text{ is the number of elements in } A.
\]
Equal likelihood

- If a sample space $S$ has $n$ elements, and all of them are equally likely, the each one has to have probability $1/n$
- What is $P(A)$ for a general set $A \subset S$?
If a sample space $S$ has $n$ elements, and all of them are equally likely, the each one has to have probability $1/n$.

What is $P(A)$ for a general set $A \subset S$?

Answer: $|A|/|S|$, where $|A|$ is the number of elements in $A$. 

Equal likelihood
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Roll two dice. What is the probability that their sum is three?
Problems

- Roll two dice. What is the probability that their sum is three?
- Toss eight coins. What is the probability that exactly five of them are heads?
Problems

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- Toss eight coins. What is the probability that exactly five of them are heads?
- In a class of 100 people with cell phone numbers, what is the probability that nobody has a number ending in 37?
Problems

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- Toss eight coins. What is the probability that exactly five of them are heads?
- In a class of 100 people with cell phone numbers, what is the probability that nobody has a number ending in 37?
- Roll ten dice. What is the probability that a 6 appears on exactly five of the dice?
Problems

▶ Roll two dice. What is the probability that their sum is three?
▶ Toss eight coins. What is the probability that exactly five of them are heads?
▶ In a class of 100 people with cell phone numbers, what is the probability that nobody has a number ending in 37?
▶ Roll ten dice. What is the probability that a 6 appears on exactly five of the dice?
▶ In a room of 23 people, what is the probability that two of them have a birthday in common?
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Recall the inclusion-exclusion identity

\[ P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \ldots + (-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1} E_{i_2} \ldots E_{i_r}) \]

\[ = + \ldots + (-1)^{n+1} P(E_1 E_2 \ldots E_n). \]
Recall the inclusion-exclusion identity

\[ P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \ldots + (-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(E_{i_1} E_{i_2} \ldots E_{i_r}) = + \ldots + (-1)^{n+1} P(E_1 E_2 \ldots E_n). \]

The notation \( \sum_{i_1 < i_2 < \ldots < i_r} \) means a sum over all of the \( \binom{n}{r} \) subsets of size \( r \) of the set \( \{1, 2, \ldots, n\} \).
Famous hat problem

- $n$ people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.
Famous hat problem

- $n$ people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.
- Inclusion-exclusion. Let $E_i$ be the event that $i$th person gets own hat.

$$P(\bigcup_{i=1}^{n} E_i) = 1 - \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r!}$$

$$\approx \frac{1}{e} \approx 0.36788$$
Famous hat problem

- $n$ people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.
- Inclusion-exclusion. Let $E_i$ be the event that $i$th person gets own hat.
- What is $P(E_1 E_2 \ldots E_r)$?

\[
\begin{align*}
\text{Answer:} & \quad \frac{(n-r)!}{n!} \\
\text{There are} & \quad \binom{n}{r} \left(\frac{(n-r)!}{n!}\right) \text{terms like that in the inclusion-exclusion sum.} \\
\text{What is} & \quad \frac{1}{r!} \\
\end{align*}
\]

\[
\begin{align*}
P(\bigcup_{1 \leq i \leq n} E_i) &= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!} \\
&\approx 1 - \frac{1}{e} \approx 0.36788
\end{align*}
\]
Famous hat problem

- $n$ people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.
- Inclusion-exclusion. Let $E_i$ be the event that $i$th person gets own hat.
- What is $P(E_1 E_2 \ldots E_r)$?
- Answer: $\frac{(n-r)!}{n!}$.

There are $\binom{n}{r}$ terms like that in the inclusion-exclusion sum.
- What is $\frac{n-r}{r!}$?
- Answer: $1$.

$P(\bigcup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!} \approx 1 / e \approx 0.36788$. 

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Famous hat problem

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- Inclusion-exclusion. Let \( E_i \) be the event that \( i \)th person gets own hat.
- What is \( P(E_{i_1}E_{i_2}\ldots E_{i_r}) \)?
- Answer: \( \frac{(n-r)!}{n!} \).
- There are \( \binom{n}{r} \) terms like that in the inclusion exclusion sum.
- What is \( \binom{n}{r} \frac{(n-r)!}{n!} \)?
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- Inclusion-exclusion. Let \(E_i\) be the event that \(i\)th person gets own hat.

- What is \(P(E_{i_1}E_{i_2}\ldots E_{i_r})\)?
  
  - Answer: \(\frac{(n-r)!}{n!}\).

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- $P(\bigcup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!}$
Famous hat problem

- $n$ people toss hats into a bin, randomly shuffle, return one hat to each person. Find probability nobody gets own hat.
- Inclusion-exclusion. Let $E_i$ be the event that $i$th person gets own hat.
- What is $P(E_1 E_2 \ldots E_r)$?
- Answer: $\frac{(n-r)!}{n!}$.
- There are $\binom{n}{r}$ terms like that in the inclusion exclusion sum. What is $\binom{n}{r} \frac{(n-r)!}{n!}$?
- Answer: $\frac{1}{r!}$.
- $P(\bigcup_{i=1}^n E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \pm \frac{1}{n!}$
- $1 - P(\bigcup_{i=1}^n E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots \pm \frac{1}{n!} \approx 1/e \approx .36788$
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What’s the probability of a full house in poker (i.e., in a five card hand, 2 have one value and three have another)?

Answer 1:

\[
\frac{\binom{5}{2} \times 13 \times 12 \times (4 \times 3 \times 2) \times (4 \times 3)}{52 \times 51 \times 50 \times 49 \times 48} = \frac{6}{4165}.
\]

Answer 2:

\[
\frac{13 \times 12 \times \binom{4}{3} \times \binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165}.
\]

What is the probability of a two-pair hand in poker?

What is the probability of a bridge hand with 3 of one suit, 3 of one suit, 2 of one suit, 5 of another suit?
What’s the probability of a full house in poker (i.e., in a five card hand, 2 have one value and three have another)?

Answer 1:

\[
\frac{\text{# ordered distinct-five-card sequences giving full house}}{\text{# ordered distinct-five-card sequences}} = \frac{\binom{5}{2} \times 13 \times 12 \times 4 \times 3 \times 2 \times 4 \times 3}{52 \times 51 \times 50 \times 49 \times 48} = \frac{6}{4165}.
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Answer 2:

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\binom{5}{2} \times 13 \times 12 \times (4 \times 3 \times 2) \times (4 \times 3) / (52 \times 51 \times 50 \times 49 \times 48) = 6/4165.
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That’s \( \binom{5}{2} \cdot 13 \cdot 12 \cdot (4 \cdot 3 \cdot 2) \cdot (4 \cdot 3) / (52 \cdot 51 \cdot 50 \cdot 49 \cdot 48) = 6/4165. \)

Answer 2:

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\frac{\text{# unordered distinct-five-card sets giving full house}}{\text{# unordered distinct-five-card sets}}
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That’s \( 13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} / \binom{52}{5} = 6/4165. \)

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That’s \(\binom{5}{2} \times 13 \times 12 \times (4 \times 3 \times 2) \times (4 \times 3)/(52 \times 51 \times 50 \times 49 \times 48) = 6/4165.\)

▶ Answer 2:

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\frac{\text{# unordered distinct-five-card sets giving full house}}{\text{# unordered distinct-five-card sets}}
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That’s \(13 \times 12 \times \binom{4}{3} \times \binom{4}{2}/\binom{52}{5} = 6/4165.\)

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