18.600: Lecture 36

Risk Neutral Probability and Black-Scholes

Scott Sheffield

MIT
Outline

Risk neutral probability

Black-Scholes
Risk neutral probability

Black-Scholes
The mathematics of today’s lecture will not go far beyond things we know.
Overview

- The mathematics of today’s lecture will not go far beyond things we know.
- Main mathematical tasks will be to compute expectations of functions of log-normal random variables (to get the Black-Scholes formula) and differentiate under an integral (to compute risk neutral density functions from option prices).

Brownian motion (as mathematically constructed by MIT professor Norbert Wiener) is a continuous time martingale. Black-Scholes theory assumes that the log of an asset price is a process called Brownian motion with drift with respect to risk neutral probability. Implies option price formula.
The mathematics of today’s lecture will not go far beyond things we know.

Main mathematical tasks will be to compute expectations of functions of log-normal random variables (to get the Black-Scholes formula) and differentiate under an integral (to compute risk neutral density functions from option prices).

Will spend time giving financial interpretations of the math.

Brownian motion (as mathematically constructed by MIT professor Norbert Wiener) is a continuous time martingale. Black-Scholes theory assumes that the log of an asset price is a process called Brownian motion with drift with respect to risk neutral probability. Implies option price formula.
The mathematics of today’s lecture will not go far beyond things we know.

Main mathematical tasks will be to compute expectations of functions of log-normal random variables (to get the Black-Scholes formula) and differentiate under an integral (to compute risk neutral density functions from option prices).

Will spend time giving financial *interpretations* of the math.

Can interpret this lecture as a sophisticated story problem, illustrating an important application of the probability we have learned in this course (involving probability axioms, expectations, cumulative distribution functions, etc.)
The mathematics of today’s lecture will not go far beyond things we know.
Main mathematical tasks will be to compute expectations of functions of log-normal random variables (to get the Black-Scholes formula) and differentiate under an integral (to compute risk neutral density functions from option prices).
Will spend time giving financial interpretations of the math.
Can interpret this lecture as a sophisticated story problem, illustrating an important application of the probability we have learned in this course (involving probability axioms, expectations, cumulative distribution functions, etc.)
Brownian motion (as mathematically constructed by MIT professor Norbert Wiener) is a continuous time martingale.
Overview

- The mathematics of today’s lecture will not go far beyond things we know.
- Main mathematical tasks will be to compute expectations of functions of log-normal random variables (to get the Black-Scholes formula) and differentiate under an integral (to compute risk neutral density functions from option prices).
- Will spend time giving financial interpretations of the math.
- Can interpret this lecture as a sophisticated story problem, illustrating an important application of the probability we have learned in this course (involving probability axioms, expectations, cumulative distribution functions, etc.)
- Brownian motion (as mathematically constructed by MIT professor Norbert Wiener) is a continuous time martingale.
- Black-Scholes theory assumes that the log of an asset price is a process called Brownian motion with drift with respect to risk neutral probability. Implies option price formula.
See desk at office hours

DESK USED BY
DR. NORBERT WIENER
AT M. I. T. 1919 - 1964
“Risk neutral probability” is a fancy term for “price probability”. (The term “price probability” is arguably more descriptive.)
“Risk neutral probability” is a fancy term for “price probability”. (The term “price probability” is arguably more descriptive.)

That is, it is a probability measure that you can deduce by looking at prices.
“Risk neutral probability” is a fancy term for “price probability”. (The term “price probability” is arguably more descriptive.)

That is, it is a probability measure that you can deduce by looking at prices.

For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?
“Risk neutral probability” is a fancy term for “price probability”. (The term “price probability” is arguably more descriptive.)

That is, it is a probability measure that you can deduce by looking at prices.

For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?

If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.
“Risk neutral probability” is a fancy term for “price probability”. (The term “price probability” is arguably more descriptive.)

That is, it is a probability measure that you can deduce by looking at prices.

For example, suppose somebody is about to shoot a free throw in basketball. What is the price in the sports betting world of a contract that pays one dollar if the shot is made?

If the answer is .75 dollars, then we say that the risk neutral probability that the shot will be made is .75.

Risk neutral probability is the probability determined by the market betting odds.
Risk neutral probability of outcomes known at fixed time $T$

- **Risk neutral probability of event $A$: $P_{RN}(A)$ denotes**

\[
\frac{\text{Price\{Contract paying 1 dollar at time $T$ if $A$ occurs \}}}{\text{Price\{Contract paying 1 dollar at time $T$ no matter what \}}}
\]

- If risk-free interest rate is constant and equal to $r$ (compounded continuously), then denominator is $e^{-rT}$.

- Assuming no arbitrage (i.e., no risk-free profit with zero upfront investment), $P_{RN}$ satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events $A_j$ are disjoint then $P_{RN}(A_1 \cup A_2 \cup ...)$ = $P_{RN}(A_1) + P_{RN}(A_2) + ...$.

- **Arbitrage example:** if $A$ and $B$ are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then we sell contracts paying 1 if $A$ occurs and 1 if $B$ occurs, buy contract paying 1 if $A \cup B$ occurs, pocket difference.
Risk neutral probability of outcomes known at fixed time $T$

- **Risk neutral probability of event $A$:** $P_{RN}(A)$ denotes
  \[
  \frac{\text{Price\{Contract paying 1 dollar at time $T$ if $A$ occurs\}}}{\text{Price\{Contract paying 1 dollar at time $T$ no matter what\}}}.
  \]

- If risk-free interest rate is constant and equal to $r$ (compounded continuously), then denominator is $e^{-rT}$.
Risk neutral probability of outcomes known at fixed time $T$

- **Risk neutral probability of event $A$:** $P_{RN}(A)$ denotes

\[
\frac{\text{Price\{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs \}}} {\text{Price\{Contract paying 1 dollar at time } T \text{ no matter what \}}}
\]

- If risk-free interest rate is constant and equal to $r$ (compounded continuously), then denominator is $e^{-rT}$.

- Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), $P_{RN}$ satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events $A_j$ are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$.
Risk neutral probability of outcomes known at fixed time $T$

- **Risk neutral probability of event $A$:** $P_{RN}(A)$ denotes

  \[
  \frac{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ if } A \text{ occurs}\}}{\text{Price}\{\text{Contract paying 1 dollar at time } T \text{ no matter what}\}}.
  \]

- If risk-free interest rate is constant and equal to $r$ (compounded continuously), then denominator is $e^{-rT}$.

- Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), $P_{RN}$ satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events $A_j$ are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$

- **Arbitrage example:** if $A$ and $B$ are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then we sell contracts paying 1 if $A$ occurs and 1 if $B$ occurs, buy contract paying 1 if $A \cup B$ occurs, pocket difference.
At first sight, one might think that $P_{RN}(A)$ describes the market’s best guess at the probability that $A$ will occur.

- Suppose $A$ is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?
  - Should be 0. Even if people think $A$ is likely, a contract paying a dollar when $A$ occurs is worthless.
- Now, suppose there are only 2 outcomes: $A$ is the event that economy booms and everyone prospers and $B$ is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think $A$ has a 0.5 chance to occur, do we expect $P_{RN}(A) > 0.5$ or $P_{RN}(A) < 0.5$?
  - Answer: $P_{RN}(A) < 0.5$. People are risk averse. In second scenario they need the money more.
At first sight, one might think that $P_{RN}(A)$ describes the market’s best guess at the probability that $A$ will occur.

But suppose $A$ is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?

Answer: $P_{RN}(A) < 0.5$. People are risk averse. In the second scenario they need the money more.
At first sight, one might think that $P_{RN}(A)$ describes the market’s best guess at the probability that $A$ will occur.

But suppose $A$ is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?

Should be 0. Even if people think $A$ is likely, a contract paying a dollar when $A$ occurs is worthless.
At first sight, one might think that $P_{RN}(A)$ describes the market’s best guess at the probability that $A$ will occur.

But suppose $A$ is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?

Should be 0. Even if people think $A$ is *likely*, a contract paying a dollar when $A$ occurs is worthless.

Now, suppose there are only 2 outcomes: $A$ is event that economy booms and everyone prospers and $B$ is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think $A$ has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?

Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.
At first sight, one might think that $P_{RN}(A)$ describes the market’s best guess at the probability that $A$ will occur.

But suppose $A$ is the event that the government is dissolved and all dollars become worthless. What is $P_{RN}(A)$?

Should be 0. Even if people think $A$ is likely, a contract paying a dollar when $A$ occurs is worthless.

Now, suppose there are only 2 outcomes: $A$ is event that economy booms and everyone prospers and $B$ is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think $A$ has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?

Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.
Suppose that $A$ is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market’s best assessment of) the probability that the Red Sox will win?

Arguably yes. The amount that people in general need or value dollars does not depend much on whether $A$ occurs (even though the financial needs of specific individuals may depend on heavily on $A$).

Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team’s prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.
Suppose that $A$ is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market’s best assessment of) the probability that the Red Sox will win?

Arguably yes. The amount that *people in general* need or value dollars does not depend much on whether $A$ occurs (even though the financial needs of specific individuals may depend on heavily on $A$).
Suppose that $A$ is the event that the Boston Red Sox win the World Series. Would we expect $P_{RN}(A)$ to represent (the market’s best assessment of) the probability that the Red Sox will win?

Arguably yes. The amount that people in general need or value dollars does not depend much on whether $A$ occurs (even though the financial needs of specific individuals may depend on heavily on $A$).

Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team’s prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.
Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).
Definition of risk neutral probability depends on choice of currency (the so-called \textit{numéraire}).

Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define $P_{RN}(A)$ to be price of contract paying this amount if and when $A$ occurs.
Definition of risk neutral probability depends on choice of currency (the so-called *numéraire*).

Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define $P_{RN}(A)$ to be price of contract paying this amount if and when $A$ occurs.

For simplicity, we focus on fixed time $T$, fixed interest rate $r$ in this lecture.
risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
- Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.
Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.

Many financial derivatives are essentially bets of this form.

Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.

Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.
Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.

Many financial derivatives are essentially bets of this form.

Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.

Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.

Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you’ll surely come out way ahead eventually.
Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.

Many financial derivatives are essentially bets of this form.

Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.

Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.

Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you’ll surely come out way ahead eventually.

Pundit: Well, you know... been busy... scruples about gambling... more to life than money...
Risk neutral probability is objective

- Check out binary prediction contracts at predictwise.com, oddschecker.com, predictit.com, etc.
- Many financial derivatives are essentially bets of this form.
- Unlike “true probability” (what does that mean?) the “risk neutral probability” is an objectively measurable price.
- Pundit: The market predictions are ridiculous. I can estimate probabilities much better than they can.
- Listener: Then why not make some bets and get rich? If your estimates are so much better, law of large numbers says you’ll surely come out way ahead eventually.
- Pundit: Well, you know... been busy... scruples about gambling... more to life than money...
- Listener: Yeah, that’s what I thought.
By assumption, the price of a contract that pays one dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.
By assumption, the price of a contract that pays one dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.

If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?

Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.

Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.

Example: if a non-divided paying stock will be worth $X$ at time $T$, then its price today should be $E_{RN}(X)e^{-rT}$.

Aside: So-called fundamental theorem of asset pricing states that (assuming no arbitrage) interest-discounted asset prices are martingales with respect to risk neutral probability. Current price of stock being $E_{RN}(X)e^{-rT}$ follows from this.
By assumption, the price of a contract that pays one dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.

If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?

Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$. 

Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.

Example: if a non-dividends paying stock will be worth $X$ at time $T$, then its price today should be $E_{RN}(X)e^{-rT}$.

Aside: So-called fundamental theorem of asset pricing states that (assuming no arbitrage) interest-discounted asset prices are martingales with respect to risk neutral probability. Current price of stock being $E_{RN}(X)e^{-rT}$ follows from this.
Prices as expectations

- By assumption, the price of a contract that pays one dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.

- If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?

  - Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.

- Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.
Prices as expectations

- By assumption, the price of a contract that pays one dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.
- If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?
  - Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.
- Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.
- Example: if a non-divided paying stock will be worth $X$ at time $T$, then its price today should be $E_{RN}(X)e^{-rT}$. 

Aside: So-called fundamental theorem of asset pricing states that (assuming no arbitrage) interest-discounted asset prices are martingales with respect to risk neutral probability. Current price of stock being $E_{RN}(X)e^{-rT}$ follows from this.
Prices as expectations

- By assumption, the price of a contract that pays one dollar at time $T$ if $A$ occurs is $P_{RN}(A)e^{-rT}$.

- If $A$ and $B$ are disjoint, what is the price of a contract that pays 2 dollars if $A$ occurs, 3 if $B$ occurs, 0 otherwise?

  Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.

- Generally, in absence of arbitrage, price of contract that pays $X$ at time $T$ should be $E_{RN}(X)e^{-rT}$ where $E_{RN}$ denotes expectation with respect to the risk neutral probability.

- Example: if a non-divided paying stock will be worth $X$ at time $T$, then its price today should be $E_{RN}(X)e^{-rT}$.

- Aside: So-called fundamental theorem of asset pricing states that (assuming no arbitrage) interest-discounted asset prices are martingales with respect to risk neutral probability. Current price of stock being $E_{RN}(X)e^{-rT}$ follows from this.
Outline

Risk neutral probability

Black-Scholes
Outline

Risk neutral probability

Black-Scholes
Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.

1997 Nobel Prize.

Assumption: the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T \sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.

Observation: $N$ normal ($\mu, T \sigma^2$) implies $E[e^N] = e^{\mu + T \sigma^2/2}$.

Observation: If $X_0$ is the current price then $X_0 = E[RN][X] e^{-rT} = E[RN][e^N] e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$.

Observation: This implies $\mu = \ln X_0 + (r - \sigma^2/2)T$.

Conclusion: If $g$ is any function then the price of a contract that pays $g(X)$ at time $T$ is $E[RN][g(X)] e^{-rT} = E[RN][g(e^N)] e^{-rT}$ where $N$ is normal with mean $\mu$ and variance $T \sigma^2$. 
More famous MIT professors: Black, Scholes, Merton.

1997 Nobel Prize.

Assumption:

\[ \log(X_T) \text{ is a normal random variable (call it } N \text{) with some known variance (call it } T \sigma^2 \text{) and some mean (call it } \mu \text{) with respect to risk neutral probability.} \]

Observation:

\[ N \text{ normal (} \mu, T \sigma^2 \text{) implies } E[e^{N}] = e^{\mu + T \sigma^2 / 2}. \]

Observation:

If \( X_0 \) is the current price then

\[ X_0 = E[RN][X] e^{-rT} = E[RN][e^{N}] e^{-rT} = e^{\mu + (\sigma^2 / 2 - r)T}. \]

Observation:

This implies \( \mu = \log(X_0) + (r - \sigma^2 / 2)T. \)

Conclusion:

If \( g \) is any function then the price of a contract that pays \( g(X_T) \) at time \( T \) is

\[ E[RN][g(X)] e^{-rT} = E[RN][g(e^{N})] e^{-rT} \text{ where } N \text{ is normal with mean } \mu \text{ and variance } T \sigma^2. \]
More famous MIT professors: Black, Scholes, Merton.

1997 Nobel Prize.

Assumption: the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T\sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.
More famous MIT professors: Black, Scholes, Merton.

1997 Nobel Prize.

**Assumption:** the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T\sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.

**Observation:** $N$ normal ($\mu$, $T\sigma^2$) implies $E[e^N] = e^{\mu + T\sigma^2/2}$.
More famous MIT professors: Black, Scholes, Merton.

1997 Nobel Prize.

**Assumption:** the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T\sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.

**Observation:** $N$ normal ($\mu$, $T\sigma^2$) implies $E[e^N] = e^{\mu + T\sigma^2/2}$.

**Observation:** If $X_0$ is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$. 

More famous MIT professors: Black, Scholes, Merton.

1997 Nobel Prize.

Assumption: The log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T\sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.

Observation: $N$ normal ($\mu, T\sigma^2$) implies $E[e^N] = e^{\mu + T\sigma^2/2}$.

Observation: If $X_0$ is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$.

Observation: This implies $\mu = \log X_0 + (r - \sigma^2/2)T$. 

Black-Scholes: main assumption and conclusion

- More famous MIT professors: Black, Scholes, Merton.
- 1997 Nobel Prize.
- **Assumption:** the log of an asset price $X$ at fixed future time $T$ is a normal random variable (call it $N$) with some known variance (call it $T\sigma^2$) and some mean (call it $\mu$) with respect to risk neutral probability.

- **Observation:** $N$ normal $(\mu, T\sigma^2)$ implies $E[e^N] = e^{\mu + T\sigma^2/2}$.

- **Observation:** If $X_0$ is the current price then $X_0 = E_{RN}[X]e^{-rT} = E_{RN}[e^N]e^{-rT} = e^{\mu + (\sigma^2/2 - r)T}$.

- **Observation:** This implies $\mu = \log X_0 + (r - \sigma^2/2)T$.

- **Conclusion:** If $g$ is any function then the price of a contract that pays $g(X)$ at time $T$ is

  $$E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$$

  where $N$ is normal with mean $\mu$ and variance $T\sigma^2$. 
A European call option on a stock at maturity date $T$, strike price $K$, gives the holder the right (but not obligation) to purchase a share of stock for $K$ dollars at time $T$.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. SS
Black-Scholes example: European call option

- A **European call option** on a stock at maturity date $T$, **strike price** $K$, gives the holder the right (but not obligation) to purchase a share of stock for $K$ dollars at time $T$.

  The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars. SS

- If $X$ is the value of the stock at $T$, then the value of the option at time $T$ is given by $g(X) = \max\{0, X - K\}$. 

18.600 Lecture 36
A **European call option** on a stock at **maturity date** $T$, **strike price** $K$, gives the holder the right (but not obligation) to purchase a share of stock for $K$ dollars at time $T$.

If $X$ is the value of the stock at $T$, then the value of the option at time $T$ is given by $g(X) = \max\{0, X - K\}$.

**Black-Scholes**: price of contract paying $g(X)$ at time $T$ is $E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$ where $N$ is normal with variance $T\sigma^2$, mean $\mu = \log X_0 + (r - \sigma^2/2)T$. 
A **European call option** on a stock at **maturity date** $T$, **strike price** $K$, gives the holder the right (but not obligation) to purchase a share of stock for $K$ dollars at time $T$.

The document gives the bearer the right to purchase one share of MSFT from me on May 31 for 35 dollars.

If $X$ is the value of the stock at $T$, then the value of the option at time $T$ is given by $g(X) = \max\{0, X - K\}$.

**Black-Scholes:** price of contract paying $g(X)$ at time $T$ is $E_{RN}[g(X)]e^{-rT} = E_{RN}[g(e^N)]e^{-rT}$ where $N$ is normal with variance $T\sigma^2$, mean $\mu = \log X_0 + (r - \sigma^2/2)T$.

Write this as

$$e^{-rT}E_{RN}[\max\{0, e^N - K\}] = e^{-rT}E_{RN}[(e^N - K)1_{N \geq \log K}]$$

$$= \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{\log K}^{\infty} e^{-\frac{(x - \mu)^2}{2T\sigma^2}} (e^x - K) \, dx.$$
The famous formula

- Let $T$ be time to maturity, $X_0$ current price of underlying asset, $K$ strike price, $r$ risk free interest rate, $\sigma$ the volatility.
Let $T$ be time to maturity, $X_0$ current price of underlying asset, $K$ strike price, $r$ risk free interest rate, $\sigma$ the volatility.

We need to compute $e^{-rT} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) dx$ where $\mu = rT + \log X_0 - T\sigma^2/2$. 
Let $T$ be time to maturity, $X_0$ current price of underlying asset, $K$ strike price, $r$ risk free interest rate, $\sigma$ the volatility.

We need to compute $e^{-rT} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) \, dx$ where $\mu = rT + \log X_0 - T\sigma^2/2$.

Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function $\Phi$. 

18.600 Lecture 36
The famous formula

- Let $T$ be time to maturity, $X_0$ current price of underlying asset, $K$ strike price, $r$ risk free interest rate, $\sigma$ the volatility.

- We need to compute $e^{-rT} \int_{\log K}^{\infty} e^{-\frac{(x-\mu)^2}{2T\sigma^2}} (e^x - K) \, dx$ where $\mu = rT + \log X_0 - T\sigma^2/2$.

- Can use complete-the-square tricks to compute the two terms explicitly in terms of standard normal cumulative distribution function $\Phi$.

- Price of European call is $\Phi(d_1)X_0 - \Phi(d_2)Ke^{-rT}$ where $d_1 = \frac{\ln\left(\frac{X_0}{K}\right) + (r + \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$ and $d_2 = \frac{\ln\left(\frac{X_0}{K}\right) + (r - \frac{\sigma^2}{2})(T)}{\sigma\sqrt{T}}$. 
If $C(K)$ is price of European call with strike price $K$ and $f = f_X$ is risk neutral probability density function for $X$ at time $T$, then $C(K) = e^{-rT} \int_{-\infty}^{\infty} f(x) \max\{0, x - K\} \, dx$. 

Differentiating under the integral, we find that $e^{rT} C'(K) = \int f(x)(x > K) \, dx = -P_{RN}\{X > K\} = F_X(K) - 1$.

We can look up $C(K)$ for a given stock symbol (say GOOG) and expiration time $T$ at cboe.com and work out approximately what $F_X$ and hence $f_X$ must be.
If $C(K)$ is price of European call with strike price $K$ and $f = f_X$ is risk neutral probability density function for $X$ at time $T$, then $C(K) = e^{-rT} \int_{-\infty}^{\infty} f(x) \max\{0, x - K\} \, dx$.

Differentiating under the integral, we find that

$$e^{rT} C'(K) = \int f(x)(-1_{x>K}) \, dx = -P_{RN}\{X > K\} = F_X(K) - 1,$$

and

$$e^{rT} C''(K) = f(K).$$
Determining risk neutral probability from call quotes

- If $C(K)$ is price of European call with strike price $K$ and $f = f_X$ is risk neutral probability density function for $X$ at time $T$, then $C(K) = e^{-rT} \int_{-\infty}^{\infty} f(x) \max\{0, x - K\} \, dx$.

- Differentiating under the integral, we find that

\[
e^{rT} C'(K) = \int f(x)(-1_{x>K}) \, dx = -P_{RN}\{X > K\} = F_X(K) - 1,
\]

\[
e^{rT} C''(K) = f(K).
\]

- We can look up $C(K)$ for a given stock symbol (say GOOG) and expiration time $T$ at.cboe.com and work out approximately what $F_X$ and hence $f_X$ must be.
Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.
Perspective: implied volatility

- Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.

- “Implied volatility” is the value of $\sigma$ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.
Risk neutral probability densities derived from call quotes are not quite lognormal in practice. Tails are too fat. Main Black-Scholes assumption is only approximately correct.

“Implied volatility” is the value of $\sigma$ that (when plugged into Black-Scholes formula along with known parameters) predicts the current market price.

If Black-Scholes were completely correct, then given a stock and an expiration date, the implied volatility would be the same for all strike prices. In practice, when the implied volatility is viewed as a function of strike price (sometimes called the “volatility smile”), it is not constant.
Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.

- **Replicating portfolio point of view:** in the simple binary tree models (or continuum Brownian models), we can transfer money back and forth between the stock and the risk free asset to ensure our wealth at time $T$ equals the option payout. Option price is required initial investment, which is risk neutral expectation of payout. "True probabilities" are irrelevant.

- **Where arguments for assumption break down:** fluctuation sizes vary from day to day. Prices can have big jumps.

- **Fixes:** variable volatility, random interest rates, Lévy jumps...
Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.

- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large $T$.
Main Black-Scholes assumption: risk neutral probability densities are lognormal.

Heuristic support for this assumption: If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large $T$.

Replicating portfolio point of view: in the simple binary tree models (or continuum Brownian models), we can transfer money back and forth between the stock and the risk free asset to ensure our wealth at time $T$ equals the option payout. Option price is required initial investment, which is risk neutral expectation of payout. “True probabilities” are irrelevant.
Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption:** risk neutral probability densities are lognormal.

- **Heuristic support for this assumption:** If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large $T$.

- **Replicating portfolio point of view:** in the simple binary tree models (or continuum Brownian models), we can transfer money back and forth between the stock and the risk free asset to ensure our wealth at time $T$ equals the option payout. Option price is required initial investment, which is risk neutral expectation of payout. “True probabilities” are irrelevant.

- **Where arguments for assumption break down:** Fluctuation sizes vary from day to day. Prices can have big jumps.
Perspective: why is Black-Scholes not exactly right?

- **Main Black-Scholes assumption**: risk neutral probability densities are lognormal.

- **Heuristic support for this assumption**: If price goes up 1 percent or down 1 percent each day (with no interest) then the risk neutral probability must be .5 for each (independently of previous days). Central limit theorem gives log normality for large $T$.

- **Replicating portfolio point of view**: in the simple binary tree models (or continuum Brownian models), we can transfer money back and forth between the stock and the risk free asset to ensure our wealth at time $T$ equals the option payout. Option price is required initial investment, which is risk neutral expectation of payout. “True probabilities” are irrelevant.

- **Where arguments for assumption break down**: Fluctuation sizes vary from day to day. Prices can have big jumps.

- **Fixes**: variable volatility, random interest rates, Lévy jumps....