18.600: Lecture 28

Lectures 17-27 Review

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Outline

Continuous random variables

Problems motivated by coin tossing

Random variable properties

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Problems motivated by coin tossing

Random variable properties

Continuous random variables

- Say X is a **continuous random variable** if there exists a **probability density function** $f = f_X$ on $\mathbb{R}$ such that $P\{X \in B\} = \int_B f(x)dx := \int 1_B(x)f(x)dx$.

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- Probability of interval $[a, b]$ is given by $\int_a^b f(x)dx$, the area under f between a and b.
- Probability of any single point is zero.
- Define **cumulative distribution function** $F(a) = F_X(a) := P\{X < a\} = P\{X \leq a\} = \int_{-\infty}^{a} f(x)dx$.

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- This formula is often useful for calculations.

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- **Gamma distribution**: time till nth event in λ Poisson point process.

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- **Expectation of geometric random variable** with parameter p is $1/p$.
- **Expectation of binomial random variable** with parameters (n, p) is np.
- **Variance of binomial random variable** with parameters (n, p) is $np(1-p) = npq$.

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- **Minimum of independent exponentials** with parameters λ_1 and λ_2 is itself exponential with parameter $\lambda_1 + \lambda_2$.

DeMoivre-Laplace Limit Theorem

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- This is $\Phi(b) - \Phi(a) = P\{a \le X \le b\}$ when X is a standard normal random variable.

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- Here $\sqrt{npq} = \sqrt{60000 \times \frac{1}{6} \times \frac{5}{6}} \approx 91.28$.
- And $200/91.28 \approx 2.19$. Answer is about $1 - \Phi(-2.19)$.

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- Values: $\Phi(-3) \approx .0013$, $\Phi(-2) \approx .023$ and $\Phi(-1) \approx .159$.
- Rule of thumb: "two thirds of time within one SD of mean, 95 percent of time within 2 SDs of mean."

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- Thus $P\{X < a\} = 1 - e^{-\lambda a}$ and $P\{X > a\} = e^{-\lambda a}$.
- Formula $P\{X > a\} = e^{-\lambda a}$ is very important in practice.
- Repeated integration by parts gives $E[X^n] = n!/\lambda^n$.

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- Thus $P\{X < a\} = 1 - e^{-\lambda a}$ and $P\{X > a\} = e^{-\lambda a}$.
- Formula $P\{X > a\} = e^{-\lambda a}$ is very important in practice.
- Repeated integration by parts gives $E[X^n] = n!/\lambda^n$.
- If $\lambda = 1$, then $E[X^n] = n!$. Value $\Gamma(n) := E[X^{n-1}]$ defined for real $n > 0$ and $\Gamma(n) = (n-1)!$.

Defining Γ distribution

- Say that random variable X has gamma distribution with parameters (α, λ) if $f_X(x) = \begin{cases} \frac{(\lambda x)^{\alpha-1} e^{-\lambda x} \lambda}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

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- Waiting time interpretation makes sense only for integer α, but distribution is defined for general positive α.

Outline

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Problems motivated by coin tossing

Random variable properties

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- And $\mathrm{Var}[X] = \mathrm{Var}[(\beta - \alpha)Y + \alpha] = \mathrm{Var}[(\beta - \alpha)Y] = (\beta - \alpha)^2 \mathrm{Var}[Y] = (\beta - \alpha)^2/12$.

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- Generally $F_Y(a) = P\{Y \leq a\} = P\{X \leq a^{1/3}\} = F_X(a^{1/3})$
- This is a general principle. If X is a continuous random variable and g is a strictly increasing function of x and $Y = g(X)$, then $F_Y(a) = F_X(g^{-1}(a))$.

Joint probability mass functions: discrete random variables

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- In general, when X and Y are jointly defined discrete random variables, we write $p(x, y) = p_{X,Y}(x, y) = P\{X = x, Y = y\}$.

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- Density: $f(x, y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F(x, y)$.

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- Latter formula makes some intuitive sense. We're integrating over the set of x, y pairs that add up to a.

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- This amounts to restricting $f(x,y)$ to the line corresponding to the given y value (and dividing by the constant that makes the integral along that line equal to 1).

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- So if $X = \max\{X_1, \ldots, X_n\}$, then what is the probability density function of X?
- Answer: $F_X(a) = \begin{cases} 0 & a < 0 \\ a^n & a \in [0,1] \\ 1 & a > 1 \end{cases}$. And $f_x(a) = F_X'(a) = na^{n-1}$.

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- So $E[X] = E[g(Y)] = \int_0^1 g(y)dy$, which is indeed the area under the graph of $1 - F_X$.

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- Since $f(x,y) = f_X(x)f_Y(y)$ this factors as $\int_{-\infty}^{\infty} h(y)f_Y(y)dy \int_{-\infty}^{\infty} g(x)f_X(x)dx = E[h(Y)]E[g(X)]$.

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- We do something similar when X and Y are continuous random variables. In that case we write $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.
- Often useful to think of sampling (X, Y) as a two-stage process. First sample Y from its marginal distribution, obtain $Y = y$ for some particular y. Then sample X from its probability distribution *given* $Y = y$.

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- In continuum setting we had $f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}$. So $E[X|Y=y]=\int_{-\infty}^{\infty} x\frac{f(x,y)}{f_Y(y)}dx$

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- Above fact breaks variance into two parts, corresponding to these two stages.

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- Can we check the formula $\mathrm{Var}(Z) = \mathrm{Var}(E[Z|X]) + E[\mathrm{Var}(Z|X)]$ in this case?

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- In other words, adding independent random variables corresponds to multiplying moment generating functions.

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- If X is exponential with parameter $\lambda > 0$ then $M_X(t) = \frac{\lambda}{\lambda - t}$.

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- Find $f_X(x) = \frac{d}{dx}F(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

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- Turns out that $E[X] = \frac{a}{a+b}$ and the mode of X is $\frac{(a-1)}{(a-1)+(b-1)}$.