18.600: Lecture 21
More continuous random variables

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Outline

Gamma distribution

Cauchy distribution

Beta distribution
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Cauchy distribution

Beta distribution
Last time we found that if $X$ is geometric with rate 1 and $n \geq 0$ then $E[X^n] = \int_0^\infty x^n e^{-x} dx = n!$. 

So $\Gamma(\alpha)$ extends the function $(\alpha-1)!$ (as defined for strictly positive integers $\alpha$) to the positive reals.

Vexing notational issue: why define $\Gamma(\alpha)$ so that $\Gamma(\alpha) = (\alpha-1)!$ instead of $\Gamma(\alpha) = \alpha!$?

At least it's kind of convenient that $\Gamma$ is defined on $(0, \infty)$ instead of $(-1, \infty)$. 

\textbf{Defining gamma function $\Gamma$}
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- This expectation $E[X^n]$ is actually well defined whenever $n > -1$. Set $\alpha = n + 1$. The following quantity is well defined for any $\alpha > 0$:
  \[ \Gamma(\alpha) := E[X^{\alpha-1}] = \int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha - 1)! \]
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Recall: geometric and negative binomials

- The sum $X$ of $n$ independent geometric random variables of parameter $p$ is negative binomial with parameter $(n, p)$. The probability $P\{X = k\}$ is given by
  
  $$(k - 1)\frac{n - 1}{p^n (1 - p)^{k - n}}.$$ 

- What's the continuous (Poisson point process) version of “waiting for the $n$th event”?
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Let’s fix a rational number $x$ and try to figure out the probability that the $n$th coin toss happens at time $x$ (i.e., on exactly $xN$th trials, assuming $xN$ is an integer).

Write $p = \lambda/N$ and $k = xN$. (Note $p = \lambda x/k$.)

For large $N$, 

$$
\left(k - 1\right)\left(k - 2\right)\ldots\left(k - n + 1\right)\frac{p^{n-1}}{(n-1)!}$$

is approximately $k^n - 1\frac{p^{n-1}}{(n-1)!}e^{-x\lambda}p = 1/N\left(\frac{\lambda x}{(n-1)!}e^{-\lambda x}\lambda\right)$.
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For large $N$, \[ \binom{k-1}{n-1} p^{n-1} (1 - p)^{k-n} p \]

\[ \approx \frac{k^{n-1}}{(n-1)!} p^{n-1} e^{-x\lambda} p = \frac{1}{N} \left( \frac{\lambda x)^{n-1} e^{-\lambda x} \lambda}{(n-1)!} \right). \]
Defining Γ distribution

The probability from previous side, \( \frac{1}{N} \left( \frac{(\lambda x)^{(n-1)} e^{-\lambda x} \lambda}{(n-1)!} \right) \) suggests the form for a continuum random variable.
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- Say that random variable $X$ has gamma distribution with parameters $(\alpha, \lambda)$ if $f_X(x) = \begin{cases} \frac{(\lambda x)^{\alpha-1} e^{-\lambda x \lambda}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$.
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- Waiting time interpretation makes sense only for integer $\alpha$, but distribution is defined for general positive $\alpha$. 
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- Beta distribution
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  \[ f(x) = \frac{1}{\pi} \frac{1}{1+x^2}. \]
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\[ F_X(x) = P\{X \leq x\} = P\{\tan \theta \leq x\} = P\{\theta \leq \tan^{-1} x\} = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} x. \]
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Find \( f_X(x) = \frac{d}{dx} F(x) = \frac{1}{\pi} \frac{1}{1+x^2} \).
The light beam travels in (randomly directed) straight line. There’s a windier random path called Brownian motion.
Cauchy distribution: Brownian motion interpretation

- The light beam travels in (randomly directed) straight line. There's a windier random path called Brownian motion.
- If you do a simple random walk on a grid and take the grid size to zero, then you get Brownian motion as a limit.

FACT: start Brownian motion at point \((x, y)\) in the upper half plane. Probability it hits negative \(x\)-axis before positive \(x\)-axis is

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\frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{y}{x}.
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Linear function of angle between positive \(x\)-axis and line through \((0, 0)\) and \((x, y)\).

Start Brownian motion at \((0, 1)\) and let \(X\) be the location of the first point on the \(x\)-axis it hits. What's \(P\{X < a\}\)?

Applying FACT, translation invariance, reflection symmetry:

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P\{X < x\} = P\{X > -x\} = \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{1}{x}.
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So \(X\) is a standard Cauchy random variable.
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- So \(X\) is a standard Cauchy random variable.
Start at (0, 2). Let $Y$ be the first point on the $x$-axis hit by Brownian motion. Again, the same probability distribution as the point hit by the flashlight trajectory.

- Flashlight point of view: $Y$ has the same law as $2X$ where $X$ is standard Cauchy.
- Brownian point of view: $Y$ has the same law as $X_1 + X_2$ where $X_1$ and $X_2$ are standard Cauchy.

But wait a minute. $\text{Var}(Y) = 4 \text{Var}(X)$ and by independence $\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 2 \text{Var}(X)$. Can this be right?

Cauchy distribution doesn't have finite variance or mean. Some standard facts we'll learn later in the course (central limit theorem, law of large numbers) don't apply to it.
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What do I mean by not knowing anything? Let’s say that I think \( p \) is equally likely to be any of the numbers \( \{0, .1, .2, .3, .4, \ldots, .9, 1\} \).
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Now imagine a multi-stage experiment where I first choose \( p \) and then I toss \( n \) coins.
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Given that number $h$ of heads is $a - 1$, and $b - 1$ tails, what’s conditional probability $p$ was a certain value $x$?
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Given that number \( h \) of heads is \( a - 1 \), and \( b - 1 \) tails, what’s conditional probability \( p \) was a certain value \( x \)?

\[
P\left(p = x | h = (a - 1)\right) = \frac{1}{\text{P}\{h = (a - 1)\}} \left(\frac{a-1}{a-1}\right)^{x-1}(1-x)^{b-1}
\]

which is \( x^{a-1}(1 - x)^{b-1} \) times a constant that doesn’t depend on \( x \).
Beta distribution

- Suppose I have a coin with a heads probability $p$ that I really don’t know anything about. Let’s say $p$ is uniform on $[0, 1]$. 

\[ B(a, b) x^{a-1} (1-x)^{b-1} \text{ on } [0, 1], \] where $B(a, b)$ is constant chosen to make integral one. Can be shown that $B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}$. 

- $E[X] = \frac{a}{a+b}$. 

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If I get, say, $a - 1$ heads and $b - 1$ tails, then what is the conditional probability density for $p$?

Turns out to be a constant (that doesn’t depend on $x$) times $x^{a-1}(1-x)^{b-1}$.

$$B(a, b) x^{a-1}(1-x)^{b-1}$$ on $[0, 1]$, where $B(a, b)$ is constant chosen to make integral one. Can be shown that $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

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▶ $\frac{1}{B(a,b)} x^{a-1}(1 - x)^{b-1}$ on $[0, 1]$, where $B(a, b)$ is constant chosen to make integral one. Can be shown that $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. 

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