18.600: Lecture 15

Poisson processes

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Outline

Poisson random variables

What should a Poisson point process be?

Poisson point process axioms

Consequences of axioms

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- General idea: if you have a large number of unlikely events that are (mostly) independent of each other, and the expected number that occur is λ, then the total number that occur should be (approximately) a Poisson random variable with parameter λ.

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- Example: number of royal flushes in a million five-card poker hands is approximately Poisson with parameter $10^6/649739 \approx 1.54$.
- Example: if a country expects 2 plane crashes in a year, then the total number might be approximately Poisson with parameter $\lambda = 2$.

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- That is, roughly a $1/e$ fraction of months has 0 foreclosures, a $1/e$ fraction has 1, a $1/(2e)$ fraction has 2, a $1/(6e)$ fraction has 3, and a $1/(24e)$ fraction has 4.

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- Joe concludes that the probability of seeing 10 foreclosures during a given month is only $1/(10!e)$. Probability to see 10 or more (an extreme *tail event* that would destroy the bank) is $\sum_{k=10}^{\infty} 1/(k!e)$, less than one in million.

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- Investors are impressed. Joe receives large bonus.

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- Let's encode this information with a function. We'd like a random function $N(t)$ that describe the number of events that occur during the first t units of time. (This could be a model for the number of plane crashes in first t years, or the number of royal flushes in first $10^6 t$ poker hands.)

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- So $N(t)$ is a **random non-decreasing integer-valued function** of t with $N(0) = 0$.
- For each t, $N(t)$ is a random variable, and the $N(t)$ are functions on the same sample space.

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- A random function $N(t)$ with these properties is a **Poisson process with rate** λ.

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- Taking limit as $n \to \infty$, can show that probability of no event in interval of length t is $e^{-\lambda t}$.
- $P\{N(t) = 0\} = e^{-\lambda t}$.
- Let T_1 be the time of the first event. Then $P\{T_1 \geq t\} = e^{-\lambda t}$. We say that T_1 is an **exponential random variable with rate** λ.

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- Axioms can be readily verified from this description.

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- Take n to infinity, and use fact that expected number of intervals with two or more points tends to zero (thus probability to see any intervals with two more points tends to zero).

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- The numbers of events occurring in disjoint intervals are independent random variables.
- Let T_k be time elapsed, since the previous event, until the kth event occurs. Then the T_k are independent random variables, each of which is exponential with parameter λ.