Binomial random variables and repeated trials

Scott Sheffield

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Outline

Bernoulli random variables

Properties: expectation and variance

More problems
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Bernoulli random variables

Properties: expectation and variance

More problems
Toss fair coin $n$ times. (Tosses are independent.) What is the probability of $k$ heads?

\[
\binom{n}{k} \frac{1}{2^n}
\]

What if coin has $p$ probability to be heads?

\[
\binom{n}{k} p^k (1-p)^{n-k}
\]

Writing $q = 1-p$, we can write this as

\[
\binom{n}{k} p^k q^{n-k}
\]

Can use binomial theorem to show probabilities sum to one:

\[
1 = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}
\]

Number of heads is binomial random variable with parameters $(n, p)$. 

Bernoulli random variables

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- Answer: $\binom{n}{k}/2^n$. 

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\text{Writing } q = 1 - p, \text{ we can write this as } \binom{n}{k} p^k q^{n-k}.
\text{Can use binomial theorem to show probabilities sum to one: } \\
1 = 1 = (p + q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.
\end{align*}$

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- Number of heads is binomial random variable with parameters \((n, p)\).
Examples

- Toss 6 fair coins. Let $X$ be number of heads you see. Then $X$ is binomial with parameters $(n, p)$ given by $(6, 1/2)$. 

- Probability mass function for $X$ can be computed using the 6th row of Pascal's triangle.

- If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal's triangle, but the probability that $X = i$ gets multiplied by $p^i (1 - p)^{n-i}$. 

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▶ If coin is biased (comes up heads with probability $p \neq 1/2$), we can still use the 6th row of Pascal’s triangle, but the probability that $X = i$ gets multiplied by $p^i(1 - p)^{n-i}$. 
Other examples

Room contains $n$ people. What is the probability that exactly $i$ of them were born on a Tuesday?

Answer: use binomial formula

$${n \choose i} p^i q^{n-i}$$

with $p = \frac{1}{7}$ and $q = \frac{6}{7}$.

Let $n = 100$. Compute the probability that nobody was born on a Tuesday.

What is the probability that exactly 15 people were born on a Tuesday?
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Let $X$ be a binomial random variable with parameters $(n, p)$. 

$\text{Expectation}$

$\text{Direct approach: by definition of expectation,}$

$\mathbb{E}[X] = \sum_{i=0}^{n} P\{X = i\} \cdot i$.

$\text{What happens if we modify the } n \text{th row of Pascal's triangle by multiplying the } i \text{ term by } i?$

$\text{For example, replace the 5th row (1, 5, 10, 10, 5, 1) by (0, 5, 20, 30, 20, 5). Does this remind us of an earlier row in the triangle?}$

$\text{Perhaps the prior row (1, 4, 6, 4, 1)?}$
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Perhaps the prior row $(1, 4, 6, 4, 1)$?
Useful Pascal’s triangle identity

Recall that \( \binom{n}{i} = \frac{n \times (n-1) \times \ldots \times (n-i+1)}{i \times (i-1) \times \ldots \times (1)} \). This implies a simple but important identity: \( i \binom{n}{i} = n \binom{n-1}{i-1} \).
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- Using this identity (and \( q = 1 - p \)), we can write

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E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i q^{n-i} = \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i q^{n-i}.
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- Rewrite this as \( E[X] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} q^{n-1-(i-1)} \).
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Rewrite this as \( E[X] = np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} q^{(n-1)-(i-1)} \).

Substitute \( j = i - 1 \) to get

\[
E[X] = np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{(n-1)-j} = np(p + q)^{n-1} = np.
\]
Decomposition approach to computing expectation

Let $X$ be a binomial random variable with parameters $(n, p)$. Here is another way to compute $E[X]$.

Think of $X$ as representing number of heads in $n$ tosses of a coin that is heads with probability $p$.

Write $X = \sum_{j=1}^{n} X_j$, where $X_j$ is 1 if the $j$th coin is heads, 0 otherwise.

In other words, $X_j$ is the number of heads (zero or one) on the $j$th toss.

Note that $E[X_j] = p \cdot 1 + (1-p) \cdot 0 = p$ for each $j$.

Conclude by additivity of expectation that $E[X] = n \sum_{j=1}^{n} E[X_j] = n \sum_{j=1}^{n} p = np$. 

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Recall identity: $i\binom{n}{i} = n\binom{n-1}{i-1}$.
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Recall identity: $i^n = n^{n-1}$.

Generally, $E[X^k]$ can be written as

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Identity gives 

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Thus $E[X^k] = npE[(Y + 1)^{k-1}]$ where $Y$ is binomial with parameters $(n - 1, p).$
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- So $\text{Var}[X] = E[X^2] - E[X]^2 = np(n - 1)p + np - (np)^2 = np(1 - p) = npq$, where $q = 1 - p$. 

Commit to memory: variance of binomial $(n, p)$ random variable is $npq$. This is $n$ times the variance you’d get with a single coin. Coincidence?
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Compute variance with decomposition trick

\[ X = \sum_{j=1}^{n} X_j, \text{ so } \]
\[ E[X^2] = E[\sum_{i=1}^{n} X_i \sum_{j=1}^{n} X_j] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j] \]
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\[ E[X_i X_j] \text{ is } p \text{ if } i = j, \text{ } p^2 \text{ otherwise.} \]
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- $E[X_i X_j]$ is $p$ if $i = j$, $p^2$ otherwise.

- $\sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i X_j]$ has $n$ terms equal to $p$ and $(n - 1)n$ terms equal to $p^2$. 

Thus $\text{Var}[X] = E[X^2] - E[X]^2 = np - np^2 = np(1 - p) = npq$. 
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- So \( E[X^2] = np + (n - 1)np^2 = np + (np)^2 - np^2 \).
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- Thus
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- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?
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- In a 100 person senate, forty people always vote for the Republicans’ position, forty people always for the Democrats’ position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?
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- An airplane seats 200, but the airline has sold 205 tickets. Each person, independently, has a .05 chance of not showing up for the flight. What is the probability that more than 200 people will show up for the flight?

- In a 100 person senate, forty people always vote for the Republicans’ position, forty people always for the Democrats’ position and 20 people just toss a coin to decide which way to vote. What is the probability that a given vote is tied?

- You invite 50 friends to a party. Each one, independently, has a 1/3 chance of showing up. What is the probability that more than 25 people will show up?