18.440 Practice Midterm: 50 minutes, 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (30 points) Twenty people in a room each have independently random birthdays among 365 possibilities. Let P be the number of pairs of people that share a birthday (i.e., the number of ways of choosing a pair of two people that share a birthday). Let T be the number ways of choosing a triple of three people that share a birthday. (If everyone has the same birthday, then $P = 20 * 19/2$ and $T = 20 * 19 * 18/6$.) Compute the following:

(a) *Write $E_{i,j}$ for event that ith and jth person have same birthday. Then $P = \sum_{i<j} 1_{E_{i,j}}$ and $E[P] = \sum_{i<j} E[1_{E_{i,j}}] = \binom{20}{2}\frac{1}{365}$.*

(b) *$Var[P] = E[P^2] - (E[P])^2$ so suffices to compute $E[P^2]$. We have*

$$E[P^2] = E[\sum_{i<j} 1_{E_{i,j}} \sum_{k<\ell} 1_{E_{k,\ell}}] = \sum_{i<j}\sum_{k<\ell} E[1_{E_{i,j}} 1_{E_{k,\ell}}].$$

The terms $E[1_{E_{i,j}} 1_{E_{k,\ell}}]$ are $\frac{1}{365}$ if $(i,j) = (k,\ell)$ and $\frac{1}{365}^2$ otherwise. There are $\binom{20}{2}$ terms of former type and $\binom{20}{2}^2 - \binom{20}{2}$ of latter, so $E[P^2] = \binom{20}{2}\frac{1}{365} + (\binom{20}{2}^2 - \binom{20}{2})\frac{1}{365^2}$.

(c) *Similar arguments to case (a) give $E[T] = \binom{20}{3}\frac{1}{365^2}$.*

(d) *We can only have $P = 5$ and $T = 1$ if we have one triple, two pairs, and 13 singletons. We count ways to do this in stages: $\binom{20}{3}$ ways to choose people to belong to triple. Then $\binom{17}{2}$ ways to choose people for first pair then $\binom{15}{2}$ ways to choose people for second pair. Given these choices, have $365!/349!$ ways to assign birthdays to each of the 16 sets. And we overcounted by a factor of 2 (since our designation of "first pair" and "second pair" is arbitrary). So the probability is*

$$\frac{\binom{20}{3}\binom{17}{2}\binom{15}{2}(365!/349!)/2}{365^{20}}.$$

(e) *We need five pairs and 10 singletons. We have $\binom{20}{2}\binom{18}{2}\binom{16}{2}\binom{14}{2}\binom{12}{2}/5!$ ways to designate the pairs (dividing by $5!$ since ordering of pairs is* *arbitrary). Given these choices, have* $365!/350!$ *ways to assign birthdays to each of the 15 sets. So the probability is*

$$\frac{(365!/350!)\binom{20}{2}\binom{18}{2}\binom{16}{2}\binom{14}{2}\binom{12}{2}/5!}{365^{20}}.$$

(f) The probability that $P = 5$ and $T => 1$ is the same as the probability that $P = 5$ and $T = 1$ (computed in (d)) since we cannot have $P = 5$ if $T \geq 2$.

2. (20 points) Compute how many:

(a) Quadruples (w, x, y, z) of non-negative integers with $w + x + y + z = 50$. *There are* $\binom{53}{50} = \binom{53}{3}$ *of these.*

(b) Ways to divide 15 books into five groups of size 1, 2, 3, 4, and 5. *There are* $\binom{15}{1,2,3,4,5}$ *of these.*

(c) "Two pair" poker hands: (i.e. 2 cards of one denomination, 2 of another distinct denomination, and one of a third distinct denomination).

3. (20 points)

(a) Roll three dice. Find the probability that there are at least two sixes given that there is at least one six. *Probability of zero sixes is* $p_0 = (5/6)^3$. *Probability of one six is* $p_1 = 3(5/6)^2(1/6)$. *Probability of two or more sixes is* $1 - p_0 - p_1$. *Answer is* $(1 - p_0 - p_1)/(1 - p_0)$.

(b) Find the conditional probability that a standard poker hand has at least 3 aces given that it has at least 2. *Just compute explicitly the probabilities* p_2, p_3, p_4 *of having* $2, 3, 4$ *aces. Answer is* $(p_3 + p_4)/(p_2 + p_3 + p_4)$.

4. (10 points) Suppose that the sample space S contains three elements $\{1, 2, 3\}$, with probabilities .5, .2, and .3 respectively. Suppose $X(s) = s^2 - 4$ for $s \in S$. Compute

(a) $\mathbb{E}X$. *Straightforward arithmetic.*

(b) $\text{Cov}(X, |X|)$. *Straightforward arithmetic.*

5. (20 points) Suppose X is Poissonian random variable with parameter $\lambda_1 = 1$, Y is an independent Poissonian random variable with $\lambda_2 = 2$, and Z is a Poissonian random variable with parameter $\lambda_3 = 3$. Assume X and Y and Z are independent and compute the following:

(a) *Trick is to note that* $X + Y + Z$ *is also Poisson with parameter* $\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 6$. *So* $P\{X + Y + Z = 8\} = e^{-\lambda}\lambda^k/k! = e^{-6}6^8/8!$.

(b) $\mathrm{Cov}(X + 2Y, 2Y + 3Z)$ *Trick is to use the bilinearity of covariance and fact that independent variables have zero covariance. We get* $Cov(X + 2Y, 2Y + 3Z) = Cov(2Y, 2Y) = Var(2Y) = 4\,Var(Y) = 4\lambda_2 = 8$.

(c) $\mathbb{E}[XYZ]$ *is (since independence implies expectation of product is product of expectations) given by* $\lambda_1\lambda_2\lambda_3 = 6$.

(d) $\mathbb{E}[X^2Y^2Z]$ *is (by same reasoning and recalling formula for second moment of Poisson random variable) given by* $(\lambda_1^2 + \lambda_1)(\lambda_2^2 + \lambda_2)\lambda_3$.